Review of *Number Theoretic Density and Logical Limit Laws* by Stanley N. Burris

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What does Schur's Tauberian theorem have to do with the fact that, in a random two-coloured linear forest, the probability that a given monadic second-order sentence holds is well-defined? The connection is the subject of a detailed theory, developed mainly by Kevin Compton over many years. The goal of this book is to explain the connection and its background, including all necessary analysis and logic, in a style accessible to a good undergraduate.

If $\mathbf{A}(x) = \sum a_n x^n$, where a_n enumerates the n-element structures in a class $\mathcal{A}$, and $\mathbf{B}(x) = \sum b_n x^n$ where b_n enumerates a subclass $\mathcal{B}$, then the *local*, *global*, and *Dirichlet density* of $\mathcal{B}$ are, respectively, the limits of

$$b(n)/a(n), \qquad \sum_{n\le x} b(n) \Big/ \sum_{n\le x} a(n), \qquad \mathbf{B}(x)/\mathbf{A}(x)$$

as $n \to \infty$, $x \to \infty$, $x \to \rho-$ respectively, where ρ is the radius of convergence of $\mathbf{A}(x)$. If $\mathbf{A}(\rho) = \infty$, each type of density extends the preceding one (in the sense that if the earlier one is defined then the later one is equal to it). The strategy for proving the existence of local density, then, falls into two parts: show that Dirichlet density exists (and is calculable), and then prove a Tauberian theorem giving necessary conditions for the local density to exist.

The results in the first six chapters are given for *additive number systems*, that is, free commutative monoids with an additive norm. (Combinatorial enumerators should think of a structure uniquely expressible as the disjoint union of connected components.) The relation between the counting sequences $p(n)$ and $a(n)$ for indecomposable (connected) and arbitrary structures respectively is the so-called *fundamental identity*

$$\sum_{n\ge 0} a(n)x^n = \prod_{n\ge 1} (1 - x^n)^{-p(n)}.$$

Sets to which the technique applies are the *partition sets*, specified by requiring that the numbers of indecomposables of various types are smaller than, equal to, or greater than a specified value.

Now the connection with logic arises thus. Let L be a finite purely relational monadic second-order language. (This means that there are no constant or function symbols, and that we are allowed to quantify over elements or subsets.) Let $\mathcal{K}$ be a class of L-structures containing the empty structure, closed under disjoint union, and having a unique decomposition into indecomposables. Then, for any L-sentence ϕ, the class of structures satisfying ϕ is a disjoint union of partition classes. So the previous results apply, and under suitable conditions we get a density law (or sometimes even a zero-one law) for L-sentences.

In the second half of the book, the corresponding analysis is given for multiplicative norms. Typically, structures are combined by direct product instead of disjoint union, power series are replaced by Dirichlet series, and Ehrenfeucht–Fraïssé games by the Feferman–Vaught theorem. A different "fundamental identity" is used, and all the analysis has to be re-done, but surprisingly similar results emerge (usually for global rather than local density). This applies to abelian groups, Heyting algebras, etc. The results are less complete, and several conjectures and open problems are discussed.

The book is clear and self-contained, and fulfils its purposes admirably. But what should you say to a student who has read the book and been hooked and wants to know more?

The methods used depend entirely on convergent series, and so only apply to counting functions with growth no faster than exponential; but there are results on faster growth, and also a variety of analytic tools available in Odlyzko's survey [3]. Burris gives an artificial counterexample to one assertion on p. 85 and asks without comment whether a "natural" example exists. If by "natural" we understand that we are counting substructures of a countably categorical structure, then much is known about growth rates (due to Macpherson in [2] and subsequent papers). Indeed, instances where a zero-one law is provided by a countable universal object are more common and fruitful than the brief note on p. 107 suggests: see [1] for examples. Other models for a random structure, like those in random graph theory, have been developed; Shelah and Spencer have spectacular results (see Winkler [4] for an accessible survey). Also there is an "inverse problem". For example, exactly half of all N-free graphs on more than one vertex are connected (since the complement of a connected N-free graph is disconnected). This simple fact determines the asymptotics completely!

References

[1] P. J. Cameron, The random graph, pp. 331–351 in *The Mathematics of Paul Erdős*, (ed. J. Nešetřil and R. L. Graham), Springer, Berlin, 1996.

[2] H. D. Macpherson, The action of an infinite permutation group on the unordered subsets of a set, *Proc. London Math. Soc.* (3) **46** (1983), 471–486.

[3] A. M. Odlyzko, Asymptotic enumeration methods, pp. 1063–1229 in *Handbook of Combinatorics* (ed. R. L. Graham, M. Grötschel, and L. Lovász), Elsevier, Amsterdam, 1995.

[4] P. M. Winkler, Random structures and zero-one laws, pp. 399–420 in *Finite and Infinite Combinatorics in Sets and Logic* (ed. N. W. Sauer, R. E. Woodrow, and B. Sands), Kluwer, Dordrecht, 1993.