

Sudoku – an alternative history

Peter J. Cameron



p.j.cameron@qmul.ac.uk

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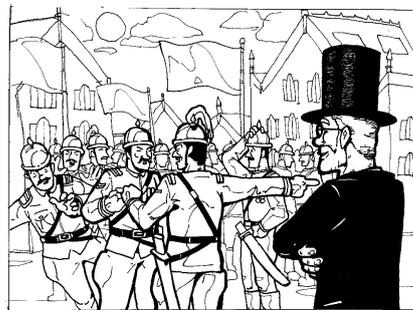
Sudoku

There's no mathematics involved. Use logic and reasoning to solve the puzzle.

Instructions in *The Independent*

But who invented Sudoku?

- Leonhard Euler
- W. U. Behrens
- John Nelder
- Howard Garns
- Robert Connelly



But he could have done it with 16 officers ...



(thanks to Liz McMahon and Gary Gordon)

Euler

Euler posed the following question in 1782.

Of 36 officers, one holds each combination of six ranks and six regiments. Can they be arranged in a 6×6 square on a parade ground, so that each rank and each regiment is represented once in each row and once in each column?

NO!!

Why was Euler interested?

A *magic square* is an $n \times n$ square containing the numbers $1, \dots, n^2$ such that all rows, columns, and diagonals have the same sum.

Magic squares have interested mathematicians for millennia, and were an active research area in the time of Arab mathematics.

Here is Dürer's *Melancholia*.



16	3	2	13
5	10	11	8
9	6	7	12
4	15	14	1

o	a	b	c
a	b	a	c
b	a	c	b
c	c	b	a

Example 1.

Euler's construction

Suppose we have a solution to Euler's problem with n^2 officers in an $n \times n$ square. Number the regiments and the ranks from 0 to $n - 1$; then each officer is represented by a 2-digit number in base n , in the range $0 \dots n^2 - 1$. Add one to get the range $1 \dots n^2$. It is easy to see that the row and column sums are constant. A bit of rearrangement usually makes the diagonal sums constant as well.

Euler called such an arrangement a *Graeco-Latin square*.

$C\beta$	$A\gamma$	$B\alpha$
$A\alpha$	$B\beta$	$C\gamma$
$B\gamma$	$C\alpha$	$A\beta$

21	01	10
00	11	22
12	20	01

8	3	4
1	5	9
6	7	2

Latin squares

A *Latin square* of order n is an $n \times n$ array containing the symbols $1, \dots, n$ such that each symbol occurs once in each row and once in each column. The name was invented by the statistician R. A. Fisher in the twentieth century, as a back-formation from "Graeco-Latin square" in the case where we have only one set of symbols.

The Cayley table of a group is a Latin square. In fact, the Cayley table of a binary system (A, \circ) is a Latin square if and only if (A, \circ) is a *quasi-group*. (This means that left and right division are uniquely defined, i.e. the equations $a \circ x = b$ and $y \circ a = b$ have unique solutions x and y for any a and b .)

About Latin squares

There is still a lot that we don't know about Latin squares.

- The number of different Latin squares of order n is not far short of n^{n^2} (but we don't know exactly). (By contrast, the number of groups of order n is at most about $n^{c(\log_2 n)^2}$, with $c = \frac{2}{27}$.)
- There is a Markov chain method to choose a random Latin square. But we don't know much about what a random Latin square looks like.
- For example, the second row is a permutation of the first; this permutation is a *derangement* (i.e. has no fixed points). Are all derangements roughly equally likely?

Orthogonal Latin squares

Two Latin squares A and B are *orthogonal* if, given any k, l , there are unique i, j such that $A_{ij} = k$ and $B_{ij} = l$.

Thus, a Graeco-Latin square is a pair of orthogonal Latin squares.

Euler was right that there do not exist orthogonal Latin squares of order 6; they exist for all other orders greater than 2.

But we don't know

- how many orthogonal pairs of Latin squares of order n there are;
- the maximum number of mutually orthogonal Latin squares of order n ;
- how to choose at random an orthogonal pair.

Projective planes

A *projective plane* is a geometry of points and lines such that any two points lie on a unique line and any two lines intersect in a unique point (together with a non-degeneracy condition to rule out trivial cases: there should exist four points with no three collinear).

A finite projective plane has $n^2 + n + 1$ points and the same number of lines, for some integer $n > 1$ called the *order* of the plane.

A projective plane of order n exists if and only if there are $n - 1$ pairwise orthogonal Latin squares of order n .

It is known that there is a projective plane of any prime power order, and that there is none of order 6 or 10. (The latter non-existence result comes from a huge computation by Clement Lam and others.)

Latin squares in cryptography

The only provably secure cipher is a *one-time pad* correctly used.

This encrypts a string of symbols in a fixed alphabet. It requires a key, a random string of the same length in the same alphabet, and an encryption table, a Latin square with rows and columns labelled by the alphabet.

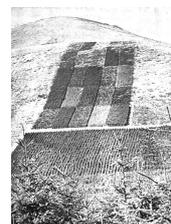
To encrypt data symbol x with key symbol y , we look in row x and column y of the encryption table, and put the symbol z in this cell in the ciphertext.

If the encryption table is not a Latin square, then either the message fails to be uniquely decipherable, or some information is leaked to the interceptor.

In the Second World War, the Japanese navy used this system with alphabet $\{0, \dots, 9\}$. Sometimes their encryption tables failed to be Latin squares.

Latin squares in statistics

Latin squares are used to “balance” treatments against systematic variations across the experimental layout.



A Latin square in Beddgelert Forest, designed by R. A. Fisher.

Behrens

The German statistician W. U. Behrens invented *gerechte designs* in 1956.

Take an $n \times n$ grid divided into n regions, with n cells in each. A *gerechte design* for this partition involves filling the cells with the numbers $1, \dots, n$ in such a way that each row, column, or region contains each of the numbers just once. So it is a special kind of Latin square.

Example 2. Suppose that there is a boggy patch in the middle of the field.

1	2	3	4	5
4	5	1	2	3
2	3	4	5	1
5	1	2	3	4
3	4	5	1	2

Nelder

The statistician John Nelder defined a *critical set* in a Latin square in 1977. This is a partial Latin square which can be completed in only one way.

A *trade* in a Latin square is a collection of entries which can be “traded” for different entries so that another Latin square is formed.

A subset of the entries of a Latin square is a *critical set* if and only if it intersects every trade.

What is the size of the smallest critical set in an $n \times n$ Latin square? It is conjectured that the answer is $\lfloor n^2/4 \rfloor$, but this is known only for $n \leq 8$.

How difficult is it to recognise a critical set, or to complete one?

Garns

It was Howard Garns, a retired architect, who put the ideas of Nelder and Behrens together and turned it into a puzzle in 1979, in *Dell Magazines*.

A Sudoku puzzle is a critical set for a gerechte design for the 9×9 grid partitioned into 3×3 subsquares. The puzzler's job is to complete the square.

Garns called his puzzle "number place". It became popular in Japan under the name "Sudoku" in 1986 and returned to the West a couple of years ago.

Connelly

Robert Connelly proposed a variant which he called *symmetric Sudoku*. The solution must be a gerechte design for all these regions:

3	5	9	2	4	8	1	6	7
4	8	1	6	7	3	5	9	2
7	2	6	9	1	5	8	3	4
8	1	4	7	3	6	9	2	5
2	6	7	1	5	9	3	4	8
5	9	3	4	8	2	6	7	1
6	7	2	5	9	1	4	8	3
9	3	5	8	2	4	7	1	6
1	4	8	3	6	7	2	5	9

Rows Columns Subsquares
Broken rows Broken columns Locations

Coordinates

We coordinatise the cells of the grid with F^4 , where F is the integers mod 3, as follows:

- the first coordinate labels large rows;
- the second coordinate labels small rows within large rows;
- the third coordinate labels large columns;
- the fourth coordinate labels small columns within large columns.

Now Connelly's regions are cosets of the following subspaces:

Rows	$x_1 = x_2 = 0$	Columns	$x_3 = x_4 = 0$
Subsquares	$x_1 = x_3 = 0$	Broken rows	$x_2 = x_3 = 0$
Broken columns	$x_1 = x_4 = 0$	Locations	$x_2 = x_4 = 0$

Affine spaces

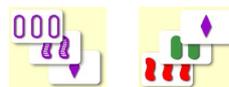
Let F be the field of integers mod 3. As we saw, the four-dimensional affine space over F has point set F^4 .

A line is the set of points satisfying three independent linear equations, or equivalently the set of points of the form $x = a + \lambda b$ for fixed $a, b \in F^4$, where λ runs through F . Note that, if $b_i = 0$, then $x_i = a_i$ for any point x , while if $b_i \neq 0$, then x_i runs through the three values in F .

Conversely, a set of three points which are either constant or take all values in each coordinate is a line.

Affine spaces and SET[®]

The card game SET has 81 cards, each of which has four attributes taking three possible values (number of symbols, shape, colour, and shading). A winning combination is a set of three cards on which either the attributes are all the same, or they are all different.



Each card has four coordinates taken from F (the integers mod 3), so the set of cards is identified with the 4-dimensional affine space. Then *the winning combinations are precisely the affine lines!*

Perfect codes

A *code* is a set C of "words" or n -tuples over a fixed alphabet F . The *Hamming distance* between two words v, w is the number of coordinates where they differ; that is, the number of errors needed to change the transmitted word v into the received word w .

A code C is *e-error-correcting* if there is *at most* one word at distance e or less from any code-word. [Equivalently, any two codewords have distance at least $2e + 1$.] We say that C is *perfect e-error-correcting* if "at most" is replaced here by "exactly".

Perfect codes and symmetric Sudoku

Take a solution to a symmetric Sudoku puzzle, and look at the set S of positions of a particular symbol s . The coordinates of the points of S have the property that any two differ in at least three places; that is, they have Hamming distance at least 3. [For, if two of these words agreed in the positions 1 and 2, then s would occur twice in a row; and similarly for the other pairs.]

Counting now shows that any element of F^4 lies at Hamming distance 1 or less from a unique element of S ; so S is a perfect 1-error-correcting code.

So a symmetric Sudoku solution is a partition of F^4 into nine perfect codes.

All symmetric Sudoku solutions

Now it can be shown that a perfect code C in F^4 is an *affine plane*, that is, a coset of a 2-dimensional subspace of F^4 . To show this, we use the *SET*[®] principle: We show that if $v, w \in C$, then the word which agrees with v and w in the positions where they agree and differs from them in the positions where they differ is again in C .

So we have to partition F^4 into nine special affine planes.

It is not hard to show that there are just two ways to do this.

One solution consists of nine cosets of a fixed subspace.

There is just one further type, consisting of six cosets of one subspace and three of another. [Take a solution of the first type, and replace three affine planes in a 3-space with a different set of three affine planes.]

All Sudoku solutions

By contrast, Jarvis and Russell showed that the number of different types of solution to ordinary Sudoku is 5 472 730 538.

They used the *Orbit-Counting Lemma*:

the number of orbits of a group on a finite set is equal to the average number of fixed points of the group elements.

An earlier computation by Felgenhauer and Jarvis gives the total number of solutions to be

6 670 903 752 021 072 936 960. Now for each conjugacy class of non-trivial symmetries of the grid, it is somewhat easier to calculate the number of fixed solutions.

Some open problems

Given a $n \times n$ grid partitioned into n regions each of size n :

- What is the computational complexity of deciding whether there exists a gerechte design?
- Assuming that there exists a gerechte design, how many are there (exactly or asymptotically), and how do we choose one uniformly at random?
- Assuming that there exists a gerechte design, what is the maximum number of pairwise orthogonal gerechte designs?
- Which gerechte designs have “good” statistical properties?

If we are given a Latin square L , and we take the regions to be the positions of symbols in L , then a gerechte design is a Latin square orthogonal to L ; so the above questions all generalise classical problems about orthogonal Latin squares.

The last two questions are particularly interesting in the case where $n = kl$ and the regions are $k \times l$ rectangles.

References

- R. A. Bailey, P. J. Cameron and R. Connelly, *Sudoku, Sudoku, gerechte designs, resolutions, affine space, spreads, reguli, and Hamming codes*, *American Math. Monthly*, to appear. Preprint available from <http://www.maths.qmul.ac.uk/~pjc/preprints/sudoku.pdf>