Scenes from mathematical life

Peter J. Cameron

Forder lectures
April 2008

Never apologize, always explain: Scenes from mathematical life

Peter J. Cameron
G. C. Steward Visiting Fellow
Gonville & Caius College
Cambridge
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There will be some mathematical content in the lectures. As Julian Havel said, mathematics is not a spectator sport; I will expect some engagement from you.

The Steward lectures

- Lecture 1: Before and beyond Sudoku

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  'I count a lot of things that there's no need to count,' Cameron said. 'Just because that's the way I am. But I count all the things that need to be counted.'

From Higman–Sims to Urysohn

Mathematicians in Scandale

My 60th birthday card (by Neill Cameron)

The adjacency matrix of a graph

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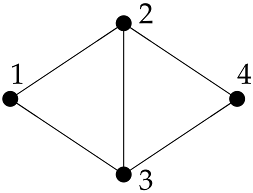

$$\begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}$$

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What does the spectrum tell us about the graph?

Graphs with least eigenvalue -2

Two classes were known:

- Line graphs (vertices of $L(\Gamma)$ are edges of Γ, joined if they meet in a vertex);
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Hoffman merged these two classes together to obtain generalized line graphs.

The theorem

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In the 1970s, Jaap Seidel (Eindhoven) and Jean-Marie Goethals (Brussels) were working on this when I visited them.

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A root system is a finite set S of non-zero vectors in real Euclidean space with the following properties:

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A root system is indecomposable if it is not contained in the union of two non-zero orthogonal subspaces; it is spherical if all roots have the same length.

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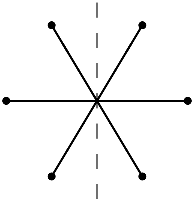
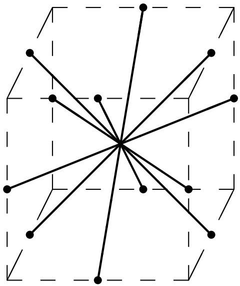
The spherical ones, which concern us here, form two infinite families, A_n (for $n \geq 1$) and D_n (for $n \geq 4$), and three "sporadic" ones, E_6, E_7 and E_8. (The subscript is the dimension of the Euclidean space.)

The root systems A_2 and A_3





The connection

Let Γ have adjacency matrix A with least eigenvalue -2. Then $2I + A$ is positive semi-definite, so is a matrix of inner products of a set of vectors in Euclidean space. The lines spanned by these vectors make angles 90° or 60° with one another.

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Enlarge this set to a maximal such set, and take vectors of fixed length in both directions along each line. *These vectors form a root system!*

So the graph can be "embedded" in A_n, D_n, E_6, E_7 or E_8.

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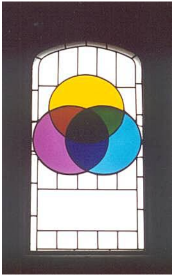
But the story is not over ...

Möbius function

This is a generalization of the “Inclusion–Exclusion Principle”.



If we know the size of the whole set, and the sizes of the circles and their intersections, we can calculate the size of the part outside all the circles. It is a sum of the other numbers multiplied by $+1$ or -1.

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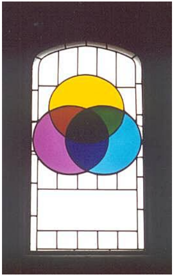

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For more general situations, we replace the ± 1s by the values of the Möbius function.

International conference on Combinatorics, Linear Algebra and Graph Colouring, at the Institute for Studies in Theoretical Physics and Mathematics (IPM) in Tehran, Iran.

IPM grounds

The mountains from IPM

Daily News

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- Competitions for students, e.g. "Discover the middle names of the invited speakers"

The winner . . .

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We have a particular permutation group acting on a set of n elements. (Actually the group $\mathrm{PSL}(2, q)$, where $n = q + 1$). We want to find, for each value of k, all possible sizes of sets of k-element subsets which admit the action of this group, and how many of each size there are.

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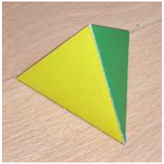
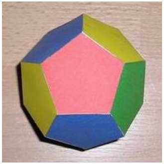
- Their orbit lengths (these are relatively easy and were worked out before).
- The so-called "Möbius function" of each possible subgroup. This turns out also to be known but is more obscure.

A trailer

There are three "exceptional" subgroups of our group, which don't fit into a regular pattern. These are the rotation groups of the regular polyhedra: tetrahedron, cube, and dodecahedron.

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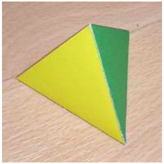  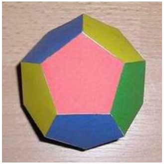

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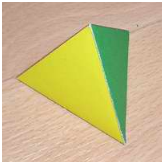

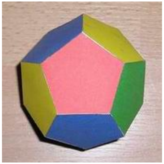

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But there is another occurrence of these numbers ...

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The number $196\,884$ is the first non-trivial Fourier coefficient of the modular function, which arises in classical (nineteenth-century) complex analysis.

At the time McKay was maybe the only mathematician in the world who knew both of these facts. This led to the conjectures termed "Monstrous moonshine" by Conway and Norton and proved by Borcherds, connections to conformal field theory and Lie algebras, etc.

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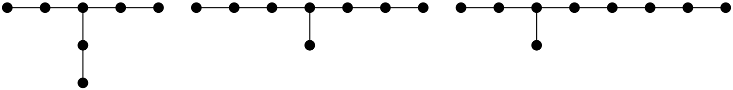
The fact that S is unitary implies that the graph is undirected. If we label each vertex with its dimension, the number at each vertex is the sum of the numbers at its neighbours.

The McKay correspondence

The graphs associated to the binary polyhedral groups are precisely the extended Coxeter–Dynkin diagrams associated with the exceptional root systems E_6, E_7 and E_8. These diagrams are obtained by taking a "fundamental basis" (with non-positive inner products), and adjoining the "largest root".



Connection numbers

Each root system spans a lattice L in Euclidean space. Because the inner products of root vectors are integers, the lattice is contained in its dual lattice $L^\dagger$, consisting of all vectors v such that $v \cdot w \in \mathbb{Z}$ for all $w \in L$. It is known that $L^\dagger/L$ is a finite group. Its order is the connection number of the root system.

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What is the connection?

From Higman–Sims to Urysohn

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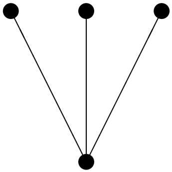

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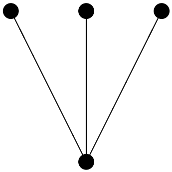

Its automorphism group has a simple subgroup of index 2, the sporadic Higman–Sims group.

Henson's graph

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For rather complicated reasons I began to wonder: Does Henson's graph admit a cyclic automorphism? That is, can you arrange the vertices along a line so that a shift one place to the right preserves the graph?

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No explicit construction of a universal sum-free set is known …

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If you choose a set S of positive integers at random, you are almost certain to get a universal set, which will give a cyclic automorphism of the random graph.

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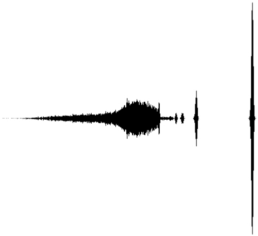

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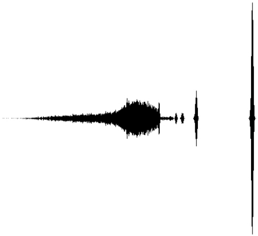

Universal sum-free sets do exist; the existence proof is non-constructive, but uses ideas from topology (Baire category) rather than probability.

The Urysohn space

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By similar methods, we were able to show that the Urysohn space also admits a cyclic isometry all of whose cycles are dense; so the space has an abelian group structure. Indeed it has many different abelian group structures! The story goes on …