Sudoku: Is it Mathematics?

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Forder lectures
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Instructions in *The Independent*

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Instructions in *The Independent*

Mathematics $\neq$ reasoning and logic???

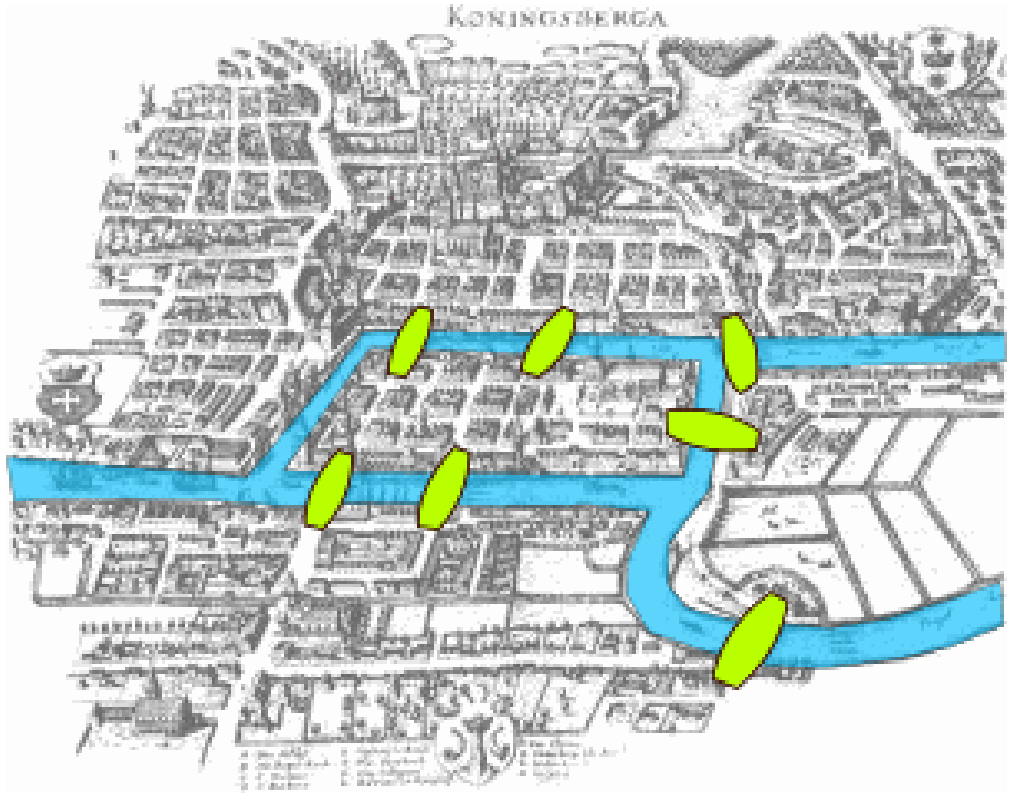
Technology transfer

To criticize mathematics for its abstraction is to miss the point entirely. Abstraction is what makes mathematics work. If you concentrate too closely on too limited an application of a mathematical idea, you rob the mathematician of his most important tools: analogy, generality, and simplicity. Mathematics is the ultimate in technology transfer.

Ian Stewart, *Does God play dice? The mathematics of chaos*, Penguin, London, 1990.

Euler

The bridges of Königsberg



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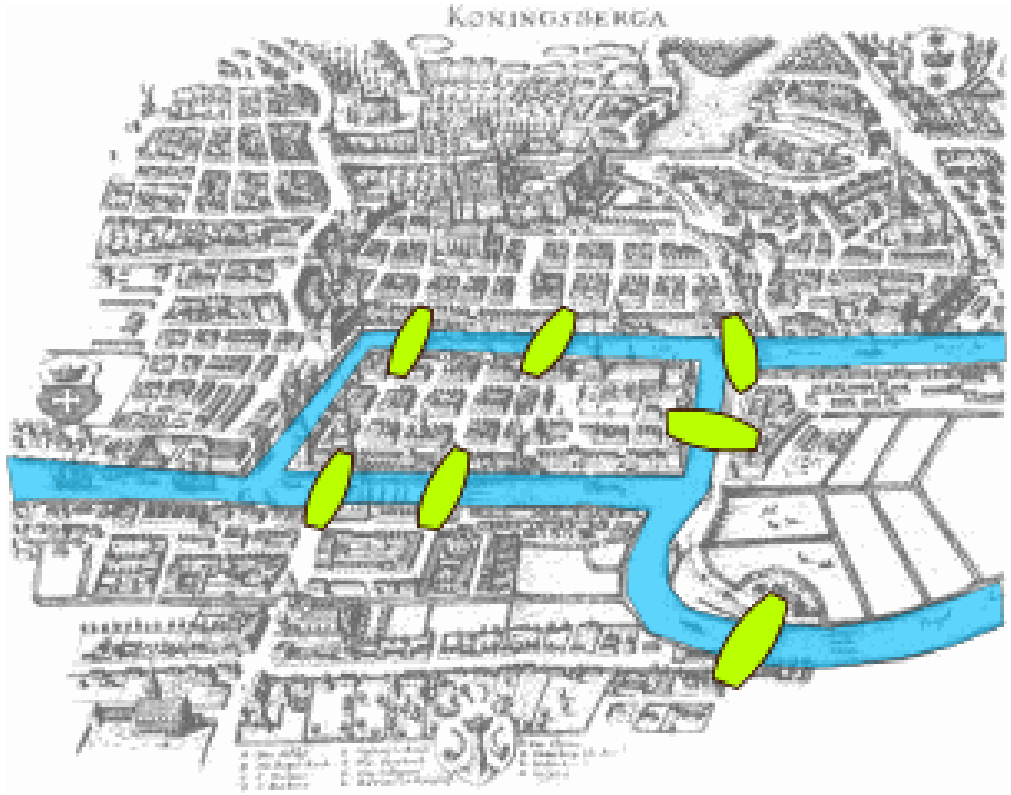

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Euler showed: No!

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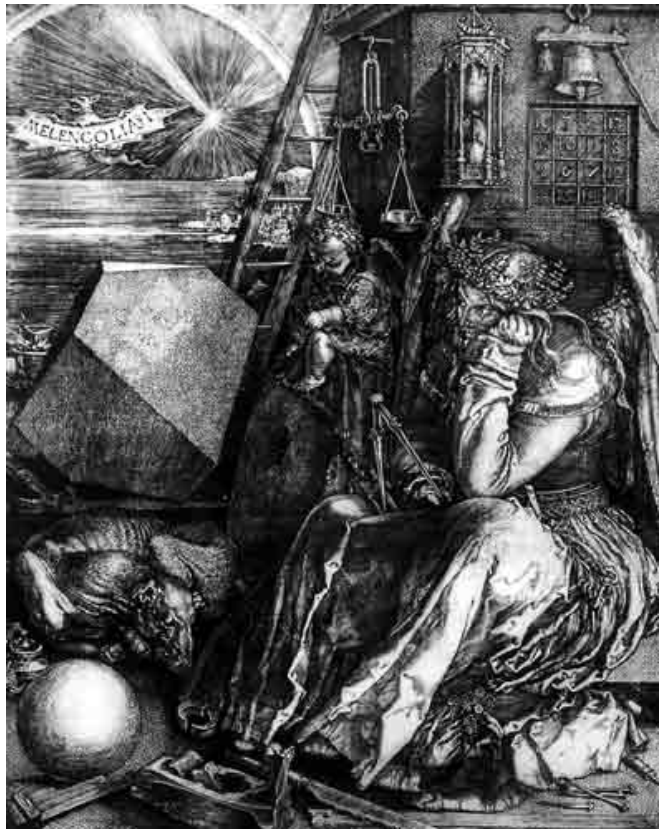
> *In the meantime, most noble Sir, you have assigned this question to the geometry of position, but I am ignorant as to what this new discipline involves, and as to which types of problem Leibniz and Wolff expected to see expressed in this way.*

Dürer's *Melancholia*



16	3	2	13
5	10	11	8
9	6	7	12
4	15	14	1

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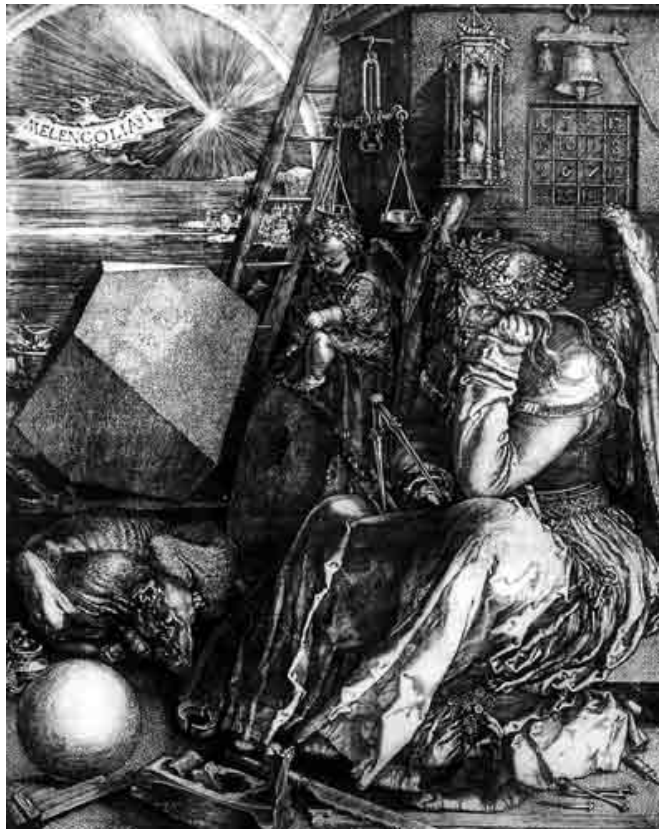

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All rows, columns, and diagonals sum to 34.

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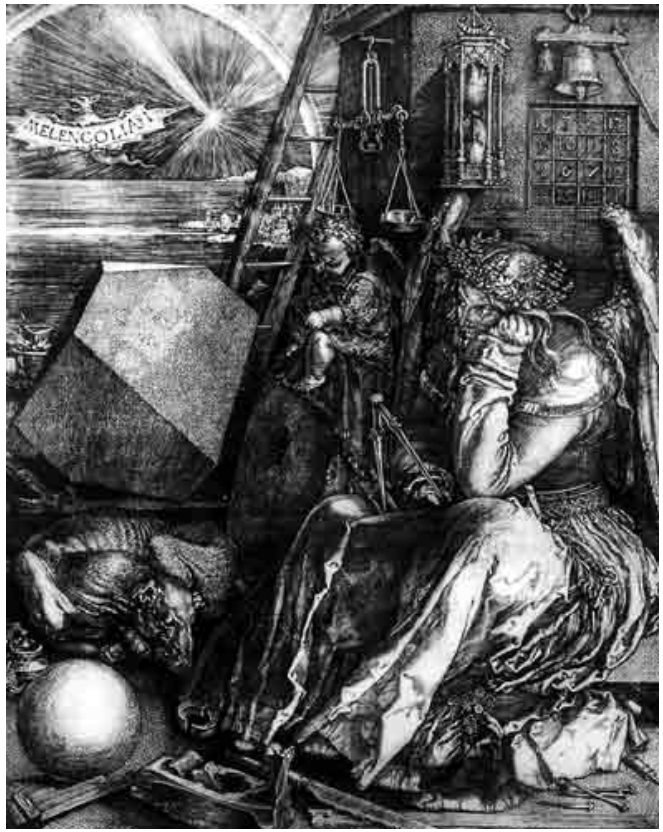

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All rows, columns, and diagonals sum to 34. The date of the picture is included in the square.

Euler's construction

Take a Graeco-Latin square of order n.

$C\beta$	$A\gamma$	$B\alpha$
$A\alpha$	$B\beta$	$C\gamma$
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So for which n do Graeco-Latin squares exist?

Euler's officers

Six different regiments have six officers, each one holding a different rank (of six different ranks altogether). Can these 36 officers be arranged in a square formation so that each row and column contains one officer of each rank and one from each regiment?

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> *Six different regiments have six officers, each one holding a different rank (of six different ranks altogether). Can these 36 officers be arranged in a square formation so that each row and column contains one officer of each rank and one from each regiment?*

Trial and error suggests the answer is "No":

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In 1900, Tarry confirmed that there is no Graeco-Latin square of order 6.

In 1960, Bose, Shrikhande and Parker showed that, apart from these two cases, Euler was wrong: Graeco-Latin squares exist for all other orders.

Latin squares

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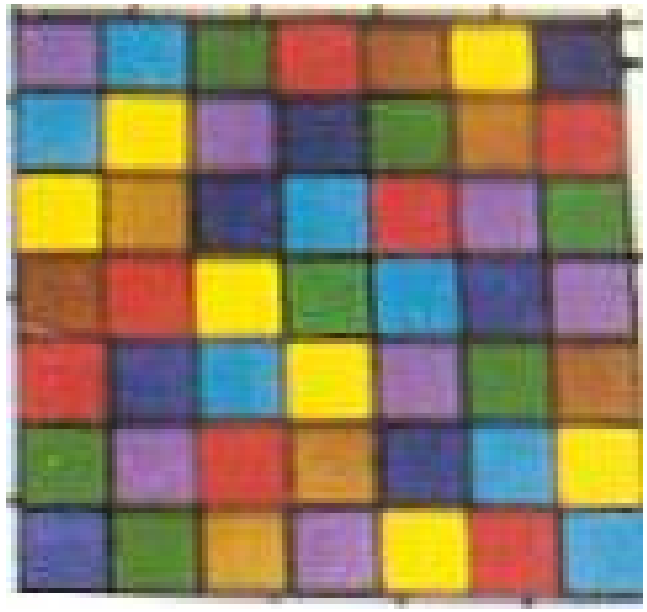
We also don't know whether there is an efficient way to decide if a given Latin square can be extended to a Graeco-Latin square.

Latin squares in statistics



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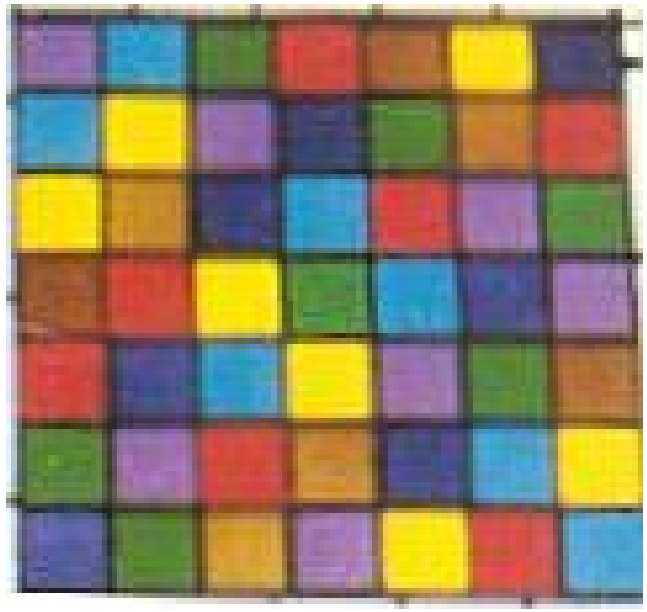

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They are useful for design of experiments in field trials where there may be spatial effects.

Latin squares in statistics

A Latin square at Rothamsted Experimental Station.

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It has the additional property of being complete: each ordered pair of distinct symbols occurs together once in a row and once in a column.

Gerechte designs

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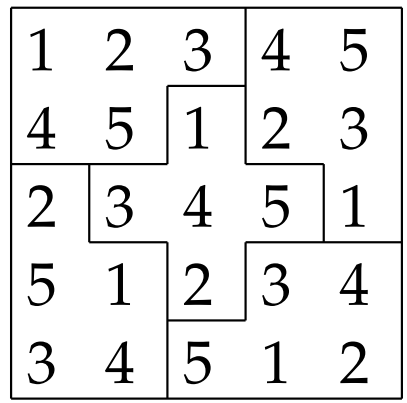

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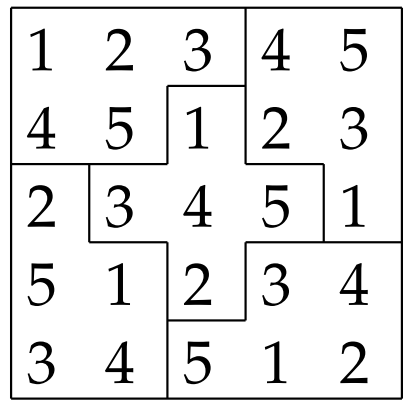

This is a gerechte design (a “fair design”).

Critical sets

John Nelder: A critical set is a partially filled Latin square which can be completed in a unique way to a Latin square, but if any entry is deleted the completion is no longer unique.

1	2		
2			
			3

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2			
			3

Critical sets were designed to study the process of "stepping" between different Latin squares by means of *trades*.

Sudoku

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New Zealander Wayne Gould popularised it in the West. The rest is history...

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This figure has been independently verified.

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We count Sudoku solutions up to

- Permuting the numbers $1, \ldots, 9$;
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This was computed by Jarvis and Russell using the Orbit-counting Lemma applied to the group $S_9 \times ((S_3 \operatorname{wr} S_3) \operatorname{wr} S_2)$ of order $9! \cdot 6^8 \cdot 2$ acting on the set of solutions counted by Felgenhauer and Jervis.

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Each number from 1 to 9 should occur once in each set of the following types:

- rows;
- columns;
- 3×3 subsquares;
- broken rows (one of these consists of three "short rows" in the same position in the three subsquares in a large column);
- broken columns (similarly defined);
- locations (a location consists of the nine cells in a given position, e.g. middle of bottom row, in each of the nine subsquares).

Example

3	5	9	2	4	8	1	6	7
4	8	1	6	7	3	5	9	2
7	2	6	9	1	5	8	3	4
8	1	4	7	3	6	9	2	5
2	6	7	1	5	9	3	4	8
5	9	3	4	8	2	6	7	1
6	7	2	5	9	1	4	8	3
9	3	5	8	2	4	7	1	6
1	4	8	3	6	7	2	5	9

Example

3	5	9	2	4	8	1	6	7
4	8	1	6	7	3	5	9	2
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8	1	4	7	3	6	9	2	5
2	6	7	1	5	9	3	4	8
5	9	3	4	8	2	6	7	1
6	7	2	5	9	1	4	8	3
9	3	5	8	2	4	7	1	6
1	4	8	3	6	7	2	5	9

Rows

Example

3	5	9	2	4	8	1	6	7
4	8	1	6	7	3	5	9	2
7	2	6	9	1	5	8	3	4
8	1	4	7	3	6	9	2	5
2	6	7	1	5	9	3	4	8
5	9	3	4	8	2	6	7	1
6	7	2	5	9	1	4	8	3
9	3	5	8	2	4	7	1	6
1	4	8	3	6	7	2	5	9

Columns

Example

3	5	9	2	4	8	1	6	7
4	8	1	6	7	3	5	9	2
7	2	6	9	1	5	8	3	4
8	1	4	7	3	6	9	2	5
2	6	7	1	5	9	3	4	8
5	9	3	4	8	2	6	7	1
6	7	2	5	9	1	4	8	3
9	3	5	8	2	4	7	1	6
1	4	8	3	6	7	2	5	9

Subsquares

Example

3	5	9	2	4	8	1	6	7
4	8	1	6	7	3	5	9	2
7	2	6	9	1	5	8	3	4
8	1	4	7	3	6	9	2	5
2	6	7	1	5	9	3	4	8
5	9	3	4	8	2	6	7	1
6	7	2	5	9	1	4	8	3
9	3	5	8	2	4	7	1	6
1	4	8	3	6	7	2	5	9

Broken rows

Example

3	5	9	2	4	8	1	6	7
4	8	1	6	7	3	5	9	2
7	2	6	9	1	5	8	3	4
8	1	4	7	3	6	9	2	5
2	6	7	1	5	9	3	4	8
5	9	3	4	8	2	6	7	1
6	7	2	5	9	1	4	8	3
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1	4	8	3	6	7	2	5	9

Broken columns

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4	8	1	6	7	3	5	9	2
7	2	6	9	1	5	8	3	4
8	1	4	7	3	6	9	2	5
2	6	7	1	5	9	3	4	8
5	9	3	4	8	2	6	7	1
6	7	2	5	9	1	4	8	3
9	3	5	8	2	4	7	1	6
1	4	8	3	6	7	2	5	9

Locations

Affine geometry

We coordinatise the cells of the grid with F^4, where F is the integers mod 3, as follows:

- the first coordinate labels large rows;
- the second coordinate labels small rows within large rows;
- the third coordinate labels large columns;
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Now the relevant regions are cosets of the following subspaces:

Rows	$x_1 = x_2 = 0$	Columns	$x_3 = x_4 = 0$
Subsquares	$x_1 = x_3 = 0$	Broken rows	$x_2 = x_3 = 0$
Broken columns	$x_1 = x_4 = 0$	Locations	$x_2 = x_4 = 0$

Affine spaces and SET

The card game SET has 81 cards, each of which has four attributes taking three possible values (number of symbols, shape, colour, and shading). A winning combination is a set of three cards on which either the attributes are all the same, or they are all different.

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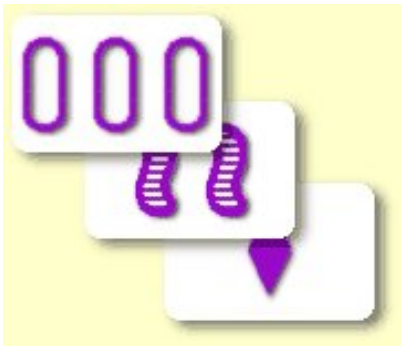

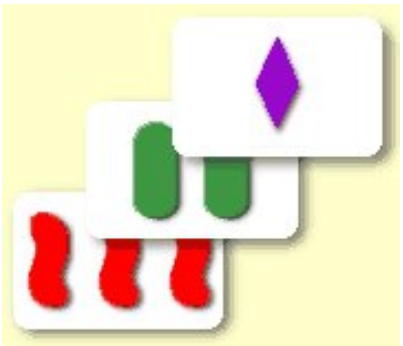

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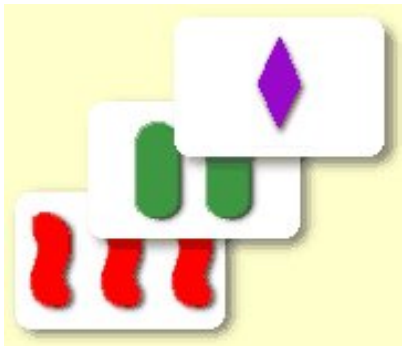

Each card has four coordinates taken from F (the integers mod 3), so the set of cards is identified with the 4-dimensional affine space. Then the winning combinations are precisely the affine lines!

Coding theory

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We transmit "words", which are strings of symbols taken from a fixed alphabet (in practice the binary alphabet $\{0, 1\}$, though any alphabet could be used). The strategy is that, instead of transmitting all possible strings, we restrict our messages to those belonging to a suitable "code". Codewords should have the property that any two of them are so different that, even if we garble one a bit, it still resembles the original more closely than it resembles any other.

An example

Alphabet $\{0, 1, 2\}$.

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$$C = \left\{ \begin{array}{lll} 0000 & 1012 & 2021 \\ 0111 & 1120 & 2102 \\ 0222 & 1201 & 2210 \end{array} \right\}.$$

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For example, the word 1221 is one step away from 1201 but at least two steps from any other codeword.

Perfect codes

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A code C is e-error-correcting if there is *at most* one word at distance e or less from any codeword. [Equivalently, any two codewords have distance at least $2e + 1$.] We say that C is perfect e-error-correcting if "at most" is replaced here by "exactly".

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The example on the last slide is a perfect code.

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- So the entire solution is a partition of the affine space into nine perfect codes.
- Using the SET test, a perfect code is an affine subspace.
- So there are only two different symmetric Sudoku solutions.

No one would doubt that this really is mathematics!

The two symmetric Sudoku solutions