3.7 Cyclic designs

Let \( \Theta = \mathbb{Z}_t \). For \( \Phi \subseteq \Theta \), a translate of \( \Phi \) is a set of the form

\[
\Phi + \theta = \{ \phi + \theta : \phi \in \Phi \}
\]

for some \( \theta \) in \( \Theta \). Of course, \( \Phi \) is a translate of itself.

It is possible to have \( \Phi + \theta_1 = \Phi + \theta_2 \) even when \( \theta_1 \neq \theta_2 \). Then \( \Phi + (\theta_1 - \theta_2) = \Phi \). Let \( l \) be the number of distinct translates of \( \Phi \): we shall abuse group-theoretic terminology slightly and refer to \( l \) as the index of \( \Phi \). Then \( \Phi + (l \mod t) = \Phi \). Moreover, \( l \) is the smallest positive integer with this property, and \( l \) divides \( t \) (for if not, the remainder on dividing \( t \) by \( l \) is a smaller positive number \( l' \) with \( \Phi + (l' \mod t) = \Phi \)).

**Definition** An incomplete-block design with treatment set \( \mathbb{Z}_t \) is a thin cyclic design if there is some subset \( \Phi \) of \( \mathbb{Z}_t \) such that the blocks are all the distinct translates of \( \Phi \): the design is said to be generated by \( \Phi \). An incomplete-block design is a cyclic design if its blocks can be partitioned into sets of blocks such that each set is a thin cyclic design.

**Example 3.13** Let \( \Phi = \{0, 1, 3\} \subset \mathbb{Z}_8 \). This has index 8, so it generates the following thin cyclic design.

\[
\{0,1,3\}, \{1,2,4\}, \{2,3,5\}, \{3,4,6\}, \{4,5,7\}, \{5,6,0\}, \{6,7,1\}, \{7,0,2\}.
\]

**Example 3.14** Here is a cyclic design for \( \mathbb{Z}_6 \) which is not thin.

\[
\{0,1,4\}, \{1,2,5\}, \{2,3,0\}, \{3,4,1\}, \{4,5,2\}, \{5,0,3\}, \{0,2,4\}, \{1,3,5\}.
\]

The index of \( \{0,1,4\} \) is 6 and the index of \( \{0,2,4\} \) is 2.

**Theorem 3.13** Let \( \Phi \subset \mathbb{Z}_t \) and let \( l \) be the index of \( \Phi \). For \( \theta \in \mathbb{Z}_t \), let

\[
m_\theta(\Phi) = |\{(\phi_1, \phi_2) \in \Phi \times \Phi : \phi_1 - \phi_2 = \theta\}|,
\]

so that

\[
\chi_\Phi \chi_{-\Phi} = \sum_{\theta \in \Theta} m_\theta(\Phi) \chi_\theta.
\]

Then, in the thin cyclic design generated by \( \Phi \),

\[
\Lambda(0, \theta) = m_\theta(\Phi) \times \frac{l}{t}
\]

and

\[
\Lambda(\eta, \zeta) = \Lambda(0, \zeta - \eta).
\]

(3.7)
3.7. CYCLIC DESIGNS

Proof Treatments 0 and \( \theta \) concur in the translate \( \Phi + \psi \) if and only if there are \( \phi_1, \phi_2 \) in \( \Phi \) such that \( \phi_1 + \psi = \theta \) and \( \phi_2 + \psi = 0 \), that is \( \psi = -\phi_2 \) and \( \theta = \phi_1 - \phi_2 \). If \( l = t \) then \( \Lambda(0, \theta) = m_0(\Phi) \). In general, the family of sets \( \Phi, \Phi + 1, \ldots, \Phi + t - 1 \) consists of \( t/l \) copies of the \( l \) distinct translates \( \Phi, \Phi + 1, \ldots, \Phi + l - 1 \), so the concurrence in the thin design is \( (t/l)m_0(\Phi) \).

Moreover, treatments 0 and \( \theta \) concur in \( \Phi + \psi \) if and only if treatments \( \eta \) and \( \eta + \theta \) concur in \( \Phi + \psi + \eta \), so \( \Lambda(0, \theta) = \Lambda(\eta, \eta + \theta) \).

**Corollary 3.14** Every cyclic design is partially balanced with respect to the cyclic association scheme on \( \mathbb{Z}_t \) defined by the blueprint \( \{0\}, \{\pm 1\}, \{\pm 2\}, \ldots \). (It may be partially balanced with respect to a cyclic association scheme with fewer associate classes.)

**Proof** Since Equation (3.7) holds in each thin component of the design, it holds overall, and

\[
\Lambda = \sum_{\theta \in \Theta} \Lambda(0, \theta)M_{\theta},
\]

where

\[
M_{\theta}(\eta, \zeta) = \begin{cases} 
1 & \text{if } \zeta - \eta = \theta \\
0 & \text{otherwise},
\end{cases}
\]

as in Section 1.4.5. But \( \Lambda \) is symmetric, so \( \Lambda(0, -\theta) = \Lambda(-\theta, 0) = \Lambda(0, \theta) \), by Equation (3.7). The adjacency matrices for the cyclic association scheme defined by the blueprint \( \{0\}, \{\pm 1\}, \{\pm 2\}, \ldots \) are \( (M_{\theta} + M_{-\theta}) \) if \( 2\theta \neq 0 \) and \( M_{\theta} \) if \( 2\theta = 0 \), so \( \Lambda \) is a linear combination of the adjacency matrices, and so the design is partially balanced with respect to this association scheme.

Suppose that \( \Delta_0, \Delta_1, \ldots, \Delta_s \) is a blueprint for \( \mathbb{Z}_t \) such that \( \Lambda(0, \theta) \) is constant \( \lambda_i \) for \( \theta \) in \( \Delta_i \). Putting \( A_i = \sum_{\theta \in \Delta_i} M_{\theta} \) gives \( \Lambda = \sum \lambda_i A_i \), and so the design is partially balanced with respect to the cyclic association scheme defined by the blueprint.

Now write \( \lambda_\theta \) for \( \Lambda(0, \theta) \).

**Technique 3.8** To calculate the concurrences in the thin design generated by \( \Phi \), form the table of differences for \( \Phi \). Try to find the coarsest blueprint such that \( \lambda_\theta \) is constant on each set in the partition.

**Example 3.13 revisited** In \( \mathbb{Z}_8 \), the block \( \{0, 1, 3\} \) gives the following table of differences.

| \( \theta \) | 0 | 1 | 3 \\
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>1</td>
<td>7</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>3</td>
<td>5</td>
<td>6</td>
<td>0</td>
</tr>
</tbody>
</table>
Therefore $\lambda_0 = 3, \lambda_1 = \lambda_2 = \lambda_3 = \lambda_5 = \lambda_6 = \lambda_7 = 1$ and $\lambda_4 = 0$. Hence the design is partially balanced for the association scheme defined by the blueprint \{0\}, \{4\}, \{1, 2, 3, 5, 6, 7\} (so this design is group divisible with groups $0 \parallel 1, 5 \parallel 2, 6 \parallel 3, 7$). □

**Definition** A subset $\Phi$ of $\mathbb{Z}_t$ is a **perfect difference set** for $\mathbb{Z}_t$ if there are integers $r, \lambda$ such that
\[
\chi_\Phi \chi_{\Phi'} = r\chi_0 + \lambda(\chi_{\mathbb{Z}_t} - \chi_0);
\]
in other words, $m_\theta(\Phi) = \lambda$ for all $\theta$ with $\theta \neq 0$.

**Proposition 3.15** The thin cyclic design generated by $\Phi$ is balanced if and only if $\Phi$ is a perfect difference set.

**Example 3.5 revisited** The subset $\{1, 2, 4\}$ is a perfect difference set for $\mathbb{Z}_7$.

Its table of differences contains every non-zero element of $\mathbb{Z}_7$ exactly once. □

**Theorem 3.16** The canonical efficiency factors of a cyclic design are
\[
1 - \frac{1}{rk} \sum_{\theta \in \mathbb{Z}_t} \lambda_\theta \eta^\theta
\]
for complex $t$-th roots of unity $\eta$ with $\eta \neq 1$.

**Proof** Use Theorems 3.12 and 2.18. □

**Technique 3.9** Let $\zeta = \exp \left( \frac{2\pi i}{t} \right)$. Then $\eta$ is a complex $t$-th root of unity if there is an integer $m$ such that $\eta = \zeta^m$. To calculate canonical efficiency factors of cyclic designs numerically, replace $\eta^\theta + \eta^{-\theta}$ by $2 \cos \left( \frac{2\pi \theta m}{t} \right)$. To calculate the harmonic mean efficiency factor $A$ as an exact rational number, leave everything in powers of $\zeta$.

**Example 3.15** Consider the thin cyclic design generated by $\{0, 1, 3, 7\}$ in $\mathbb{Z}_9$.

\[
\begin{array}{c|c|c|c}
0 & 1 & 3 & 7 \\
\hline
0 & 0 & 1 & 3 \\
1 & 8 & 0 & 2 \\
3 & 6 & 7 & 0 \\
7 & 2 & 3 & 5 \\
\end{array}
\]
Thus the eigenvalues of $\Lambda$ are

$$4 + (\eta + \eta^{-1}) + 2(\eta^2 + \eta^{-2}) + 2(\eta^3 + \eta^{-3}) + (\eta + \eta^{-4})$$

where $\eta^9 = 1$. If $\eta^3 = 1$ and $\eta \neq 1$ then $\eta + \eta^{-1} = -1$ (the cube roots of unity sum to zero) so the eigenvalue is

$$4 - 1 - 2 + 4 - 1 = 4;$$

otherwise it is

$$4 + \eta^2 + \eta^{-2} - 2 = 2 + \eta^2 + \eta^{-2},$$

because the primitive ninth roots of unity sum to zero (because all the ninth roots do). Let $\zeta$ be a fixed primitive ninth root of unity, and put $x = \zeta + \zeta^{-1}, y = \zeta^2 + \zeta^{-2}$ and $z = \zeta^4 + \zeta^{-4}$. Then the canonical efficiency factors are

$$\frac{3}{14}, \frac{14 - x}{16}, \frac{14 - y}{16}, \frac{14 - z}{16},$$

all with multiplicity 2.

Substituting $x = 2 \cos 40^\circ, y = 2 \cos 80^\circ, z = 2 \cos 160^\circ$ gives

$$0.7500, \ 0.7792, \ 0.8533 \ and \ 0.9925$$

to 4 decimal places, and $A = 0.8340$.

To do the exact calculation, we note first that $x + y + z = 0$. Then

$$xy = (\zeta + \zeta^{-1})(\zeta^2 + \zeta^{-2})$$

$$= \zeta + \zeta^3 + \zeta^{-3} + \zeta^{-1}$$

$$= x - 1,$$

and similarly $yz = y - 1$ and $zx = z - 1$. Therefore $xy + yz + zx = x + y + z - 3 = -3$

and $xyz = (x - 1)z = xz - z = z - 1 - z = -1$.

Now

$$\frac{1}{14 - x} + \frac{1}{14 - y} + \frac{1}{14 - z}$$

$$= \frac{(14 - x)(14 - y) + (14 - x)(14 - z) + (14 - y)(14 - z)}{(14 - x)(14 - y)(14 - z)}$$

$$= \frac{3 \cdot 14^2 - 28(x + y + z) + (xy + yz + zx)}{14^3 - 14^2(x + y + z) + 14(xy + yz + zx) - xyz}$$

$$= \frac{3 \cdot 14^2 - 3}{14^3 - 3 \cdot 14 + 1} = \frac{195}{901}.$$
so

\[ 4A^{-1} = \frac{4}{3} + \frac{16 \times 195}{901} \]

so

\[ A^{-1} = \frac{1}{3} + \frac{4 \times 195}{901} = \frac{3241}{2703} \]

and

\[ A = \frac{2703}{3241}. \]