Computing in the Fischer–Griess Monster
Individual Grant Review
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A Background

The theory of finite groups is of central importance in mathematics, and finds wide applications in all the physical sciences and elsewhere. In essence it is the deep study of symmetry in all its innumerable manifestations, and so has applications in all situations where symmetry occurs. The building blocks of finite groups are the 'simple' groups, analogous to the prime numbers in number theory, which are so-called because they cannot be broken down into smaller pieces. The study of 'simple' groups, however, just like the study of prime numbers, turns out to be not simple at all.

The completion around 1980 of the massive world-wide project for complete classification of the finite simple groups (see, for example, [17]), revealed that they mostly fall into various reasonably well understood families, *with precisely twenty-six exceptions*, known as the 'sporadic' simple groups. These range in size from the little Mathieu group, known since 1860, which has 7920 elements, to the Fischer–Griess Monster (or Friendly Giant), ([18], [12]), suspected since 1973 but not constructed until 1980, which has nearly 10^{54} elements. The latter is of particular interest, and all but six of the other sporadic groups may be found within it. It has attracted enormous interest since its discovery and turns out to be connected with such diverse areas as modular forms, quantum field theory and Kac–Moody algebras [4]. The recent award of a Fields Medal to Richard Borcherds was for his work in this area. At least two conferences have been devoted entirely to this single group and its properties.

Now that we know the names of all the finite simple groups, attention has shifted to studying their properties, especially their maximal subgroups, and Brauer character tables. There has been a major worldwide project in classifying the maximal subgroups of the 'generic' simple groups, which is still very much in progress. When it comes to the sporadic groups, however, the 'generic' methods do not apply, and essentially only 'ad hoc' methods are useful.

Computation has played, and continues to play, an important role in the investigation of the other 25 sporadic simple groups (see, for example, the works of the proposer, P. Kleidman and S. A. Linton, referred to in the bibliography in [30]). Such computation generally depends on representing elements of the group as permutations or matrices and determining the images of points or vectors under group elements and products of group elements. With modern computers, one can work with matrices up to dimension 20000 or so, over small fields, and with permutations on up to 10 million points. Unfortunately, the Monster has no matrix representation smaller than dimension 196882, or permutation representation on fewer than about 10^{20} points. For this reason, and since most of the relevant algorithms are cubic in the dimension, existing methods will not extend directly to this group.

In this project an alternative, more mathematically sophisticated approach is taken. We no longer store the group generators as huge matrices, using matrix multiplication to find the images of vectors under group elements. Instead we store the generators as subroutines which compute the images of vectors directly. This approach makes certain calculations enormously faster, but at the same time necessitates a complete rethinking of the methods for solving standard problems. Essentially this is because we can no longer multiply group elements together, so that algorithms which in the standard model are polynomial-time, now become exponential-time.

B Key advances and supporting methodology

B.1 Aims and Objectives

The aim of the project was to further research into and understanding of:

- the Fischer–Griess Monster;
- sporadic (and other) simple groups and related structures and
- computational algebra.

The specific objectives of the project were:

- to improve existing software enabling elements of the Fischer–Griess Monster to be represented and manipulated, so that it can usefully be made available to other researchers in the field;
- to make progress in the maximal subgroup problem for the Monster—specifically we aim to deal with at least half of the remaining 11 cases;
- to gain insight into the Monster, in particular into its subgroup structure; and
- to answer other specific questions about the Monster, which other researchers in the field wish to know the answers to.

These objectives were achieved:

- Software was improved and converted into both GAP and MAGMA. The latter version is by far the faster of the two, and will be made available in a future version of MAGMA.
- Six of the 11 cases of the maximal subgroup problem have been completed, and three more are in progress, with completion envisaged soon.
- Much other information about subgroups of the Monster was obtained.
- Work is in progress on various aspects: classification of 'nets', character tables of some maximal subgroups, explicit constructions of small representations of some maximal subgroups, etc.

B.2 Methodology

Experience over many years has shown that the only way to determine completely the very small maximal subgroups of a large sporadic group is by extensive computer searches, coupled with detailed theoretical knowledge of the larger and already known maximal subgroups. The basic method is to build up from small known subgroups (usually soluble groups and A_5) by extending the normalizer of some proper subgroup. On the other hand, this group is still so big relative to available technology, that a naïve application of standard techniques will not work. Our work uses a novel approach, avoiding multiplications of group elements entirely. This entails quite new methods of computing with elements of the group, and therefore new algorithms for finding the necessary subgroups.

In the case of the Monster, we have three computer constructions, the 2-local construction over $GF(3)$ (see [31]), and the 3-local construction over $GF(2)$ (see [26]) and over $GF(7)$. For classifying maximal subgroups, we have so far used the first construction exclusively. The extra benefits of working with involutions seem to be crucial for our calculations.

The two constructions using the 3-local subgroups were used to calculate character values modulo 14, in order to determine conjugacy classes of elements. The subgroup $3^{1+12}2{\cdot}Suz{:}2$ has been constructed again over the field of order 103, in order to help in calculating the character table of this group.

One crucial step in our work (for the cases $L_2(16)$ and $U_3(4)$ of the maximal subgroup problem) is to create a small representation of the subquotient $C(5B)/\langle 5B\rangle \cong 5^6{:}2{\cdot}J_2$ from specific generators of $C(5B)$ in the 196882-dimensional representation of the Monster. We used the techniques of Lübeck and Neunhöffer [32] to create such a representation, implicitly permuting an orbit of 15625 vectors (which is too big to be stored in our computer memory).

B.3 Description of the work undertaken

The 11 cases of the maximal subgroup problem which were left open at the start of the grant were the classifications of subgroups with socle isomorphic to $L_2(q)$ ($q = 7, 8, 13, 16, 17, 27$), $U_3(q)$ ($q = 3, 4, 8$), $L_3(3)$ and $Sz(8)$. In each case we needed to identify a suitable generating amalgam, such that it is possible to enumerate all realizations of the amalgam in the Monster. The first amalgam we looked at was $(S_4, S_4)_{S_3}$ (it turned out that we did not need to use $(S_4, S_4)_{D_8}$). This dealt with the cases $L_2(7)$, $L_2(17)$, $L_3(3)$ and $U_3(3)$ (see [20]). There were several different classes of S_4 to consider, and the

CPU requirement is measured in decades for each case. As we were compelled to continue borrowing computing resources for these cases, they took longer than originally anticipated.

The next case we dealt with was $L_2(13)$, via the amalgam $(13{:}6, D_{12})_6$. The number of cases here is huge, and the computations are still continuing, although all the programming is complete and we simply have to wait for the programs to finish running.

Two more cases have been completed, namely $L_2(8)$ and $L_2(16)$. The former used the amalgam $(2^3{:}7, D_{14})_7$, and the latter $(A_5, D_{30})_{D_{10}}$. Work is still in progress on the cases $U_3(4)$ and $U_3(8)$, and we envisage that these cases will be completed soon. The case $U_3(4)$ is being done by taking a subgroup $A_5 \times 5$ (which in the case we are looking at, when the A_5 contains $5B$-elements, is unique up to conjugacy in the Monster), and extending a diagonal 5-element to D_{10}. This merely needs more computer time to run through the cases which have already been determined. The case $U_3(8)$ is less far advanced, but we believe that the amalgam $(L_2(8), L_2(8))_{2^3}$ will work here.

The remaining two cases, $L_2(27)$ and $Sz(8)$, have so far remained resistant to our attacks. In these cases there is no convenient amalgam to use, and we have not made as much progress as we might have hoped in utilising an inconvenient amalgam! (The amalgam $(2^3{:}7, D_{14})_7$ for $Sz(8)$ unfortunately does not work. In the case of $L_2(8)$, we can complete the calculation because the 7-elements are in class $7B$, and we can show that the group $2^3{:}7$ lies inside the involution centralizer. For $Sz(8)$ this strategy does not work because we have to consider $7A$-elements as well.)

We also calculated the Schur indicator of the 196882-dimensional representation over $GF(2)$. This turns out to be $+$, showing that the Monster lies inside the corresponding orthogonal group [23].

At the same time, my research student Richard Barraclough was working on various aspects of the structure of the Monster. First, he improved the software for both the 3-local constructions, so that he could measure traces of matrices mod 2 and mod 7 effectively, thereby determining character values modulo 14. Combining this with the order of the element, and the same information for various powers, gives enough information to determine the conjugacy class to which the element belongs, up to some ambiguities [3].

Secondly, he is looking at classifying 'nets' in the sense of S. P. Norton. These are obtained by applying braiding operations to triples (a, b, c) of $2A$-elements in the Monster. These nets have 'faces' with at most 6 sides, so are either genus 0 ('footballs') or genus 1 (toroidal). So far he has calculated all the nets centralized by an element of prime order ≥ 7. Work is in progress on those centralized by elements of order 2, 3 or 5, after which only those with trivial centralizer remain. These of course are the hardest, and most interesting, cases.

In the course of this work he has calculated the character table of $3^{1+12}2Suz2$, and the class fusion to the Monster, in order to identify all the nets centralized by an element of class $3B$.

When Dr Holmes took up her Dorothy Hodgkin Fellowship, Dr Bray took over her post for the final 8 months of the project. I decided to utilise his expertise in cohomology and non-split extensions of groups, and characteristic 0 representations, to investigate other interesting aspects of the Monster. First, we constructed representations of most of the (known) maximal subgroups of the Monster. The interesting cases here are the p-local subgroups, especially when p is small. He used his knowledge to excellent effect in constructing groups such as the non-split $3^8O_8^-(3).2$ in 204 dimensions over $GF(3)$, independently of the existence of the Monster.

The only maximal subgroups which have not been treated in this way are the 2-local subgroups $2.B$, $2^{1+24}Co_1$, $2^{10+16}O_{10}^+(2)$, $2^{5+10+20}(S_3 \times L_5(2))$ and $2^{3+6+12+18}(L_3(2) \times 3S_6)$.

Secondly, Dr Bray initiated a construction of the Monster representation in characteristic 0. This has not yet been completed, and may not be of much use unless or until we have better methods of computing with it.

C Project plan review

The proposed start date of September 2002 was brought forward, as the proposed PDRA, Dr Holmes, was available earlier, and we started instead in May 2002. Purchase and installation of the computer was however delayed by some months because the School computer officer did not have the time to deal with it. Other than this, the project ran closely according to plan.

The transfer of Dr Holmes onto other funding after 16 months enabled us to appoint Dr Bray for 8 months to work on other aspects of the Monster.

Our original plan to convert our software into a GAP share package was abandoned, as the GAP infrastructure was inadequate to support the computations required. Instead it is to be incorporated into a future version of MAGMA. Papers have been prepared for publication during the course of the project, rather than at the end, as this seems more efficient. Timing of visits to collaborators and conferences was changed according to availability.

D Research impact and benefits to society

Mathematicians working with groups greatly appreciate the ability to perform explicit calculations, as the wide use of packages such as GAP and MAGMA testifies. Similarly, the explicit generators for the other 25 of the 26 sporadic simple groups which I have collected (see [37]) are well used. This suggests that there will be a strong demand for our explicit generators for the Monster, and the MAGMA programs to manipulate them. This will give an opportunity for mathematicians who work with the other sporadic groups to extend their work to the Monster.

Other immediate beneficiaries of this work will be the considerable number of mathematicians who study the Monster and the structures associated to it. For example, Jianbei An (Auckland) is using our constructions of maximal subgroups to verify the Alperin–McKay–Dade–Uno conjectures for the Monster (see also [1, 2]). The Monster has a significance beyond being the largest of the sporadic simple groups, due to its connections with other areas of mathematics and theoretical physics, including modular functions, and string theory, so researchers in these other areas will also benefit.

The work on maximal subgroups is a significant contribution to the ongoing world-wide project on classifying the maximal subgroups of all finite simple groups. Once the remaining cases in the Monster are completed (probably in the next couple of years) this will complete the determination of all the maximal subgroups of all the sporadic simple groups, which will be a siginificant milestone.

E Explanation of expenditure

Expenditure closely followed the original plan, and paid for (i) a PDRA to do the work, and (ii) a powerful computer to perform the necessary calculations.

Dr Petra Holmes was appointed at the start of the grant, as envisaged in the application, before leaving after 16 months to take up a Dorothy Hodgkin Research Fellowship in her own right. She was replaced for the final 8 months of the grant by Dr John Bray.

The computer which we bought was a slight upgrade of what was proposed in the application, made possible by continued decreases in the prices of equipment. In addition we were able to buy a workstation for the RA, a printer, and other small items.

Travel money was used to fund (or partly fund) various visits to collaborators: the MAGMA group in Sydney (by Bray and Wilson), Linton in St Andrews (by Holmes), O'Brien and An in Auckland (by Wilson), the Cambridge group (by Holmes), and the Aachen group (by Holmes and Wilson). The visits to Cambridge, Aachen and Auckland were combined with attendance at conferences, and additionally Holmes and Wilson were funded to attend the international Computational group theory conference in Columbus, Ohio, in March 2003.

Unfortunately, the sum requested for computer officer support was removed by EPSRC from the grant, which meant we were unable to pay for this essential service. This had a serious knock-on effect, as it delayed the commissioning of our computational facility by several months.

F Further research or dissemination activities

F.1 Other research carried out using our equipment

A significant benefit of the grant was that the high-powered computing equipment was available for use by other researchers when we did not require sole use of it. Our nodes were added to an existing cluster, to form a cluster of 20 dual-processor nodes of varying specifications. Many people used this facility, including:

- Gerhard Röhrle and Simon Goodwin: Studied the action of a parabolic subgroup P of a simple algebraic group on the Lie algebra of its unipotent radical. They classified all submodules which are prehomogeneous spaces, when P is either a Borel subgroup in rank at most 8, or an arbitrary parabolic subgroup in types E_6 and F_4, subject to certain conditions.

- Simon Nickerson: Finding words in standard generators for the maximal subgroups of HN.2, O'N.2, $Fi_{22}.2$ and Fi_{24} [33]. Computing explicit complex representations for finite groups of Lie type (in progress).
- Gary Sharpe and Simon Watt: The cluster was utilised in our work on the initiation and stability of cylindrically and spherically expanding detonation waves. This was acheived by using the cluster for high-resolution calculations using the CFD package, Cobra [36].

F.2 Other work carried out by the PDRAs

Another significant benefit of the grant was that the two PDRAs were able to contribute to other research projects. In particular, Dr Bray has continued work related to an earlier EPSRC grant (GR/N27491/01), completing work on symmetric presentations of the Fischer groups [6] and making further improvements to his ground-breaking Double Coset Enumerator [5]. We have also proved some new results on automorphism groups of finite groups by constructing various families of finite groups with unexpectedly small automorphism groups [8, 9], and collaborated with J. S. Wilson in proving a new characterisation of finite soluble groups [7]. In addition, Dr Bray has made contributions to the Web-Atlas of Representations [37], anticipating his appointment on the grant GR/S41319/01.

Dr Holmes has proved results on minimal factorisations of some sporadic simple groups [21, 25], (which has applications to cryptography), and on 'coverings' of some of these groups (as a union of subgroups) [22], which has applications in combinatorics.

F.3 Dissemination and exploitation

The results of the project were and will be disseminated through the usual academic routes of publication in refereed journals and presentation at national and international conferences. By the time that all the work on this project has been written up, we would expect to publish approximately 10 papers that would not have been written had this project not been funded. In addition, the grant has greatly facilitated the writing of many other papers not directly related to the grant project itself. We have already given seminars on this work in Perth (Australia), and at international conferences in Columbus (Ohio) and Edinburgh.

In addition the data and software produced will be made available on the Internet, and we anticipate that the software we have produced will be available in MAGMA v2.12.

F.4 Further work

As indicated above, the project is continuing in a number of directions, some of which may be the subject of future grant applications. Work on the maximal subgroup problem is being continued by Dr Holmes in her new Research Fellowship. The work of Dr Bray on the Monster and other groups is continuing under the auspices of my new EPSRC grant GR/41319/01, 'A world-wide-web atlas of group representations', as the latter can subsume many different calculations of group-theoretical data.

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