MSM120—1M1
First year mathematics for civil engineers
Revision notes 2

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Introduction to differentiation The slope (or gradient) of a straight line can be defined as the ratio of the change in the y-value to the change in the x-value. So if the line has equation $y = mx + c$, then every unit change in x corresponds to a change of magnitude m in y, and the slope is m.

For curves, however, this simple definition will not work, as the slope varies as x and y vary. To get a sensible definition of the slope, we need to look at a very small change in x, and see what the corresponding small change in y is. Mathematically, we then take the 'limit', as these changes become smaller and smaller (i.e. as they 'tend to zero').

Let us write δx for a small change in x—that is, we imagine x changing from x to $x + \delta x$. At the same time, the value of y changes from y to $y + \delta y$, and the slope is approximately $\frac{\delta y}{\delta x}$. Our whole problem now is to calculate δy, given δx. [Warning: δx does NOT mean $\delta \times x$, it is a single concept, sometimes written δ_x to make this clear.]

To take an example, let $y = x^2$. Then $y + \delta y$ is the value of y when x has changed to $x + \delta x$, that is

$$y + \delta y = (x + \delta x)^2 = x^2 + 2x.\delta x + (\delta x)^2.$$

Subtracting the equation $y = x^2$ gives us

$$\begin{aligned} \delta y &= 2x.\delta x + (\delta x)^2 \\ \frac{\delta y}{\delta x} &= 2x + \delta x \end{aligned}$$

Now as δx tends to 0, the last term disappears, and in the limit we obtain a slope of $2x$. We express this as

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 2x.$$

Formally, then, the definition of the slope $\frac{\mathrm{d}y}{\mathrm{d}x}$ is as the limit of $\frac{\delta y}{\delta x}$ as δx tends to 0.

Let us look now at some more examples of differentiation, using the definition in terms of δy and δx (that is, 'from first principles'). If $y = x^n$, for n a positive integer, we can use the binomial theorem to show that

$$\begin{aligned} y + \delta y &= (x + \delta x)^n \\ &= x^n + n.\delta x.x^{n-1} + \frac{n(n-1)}{2!}.(\delta x)^2.x^{n-2} + \cdots \\ &= y + n.\delta x.x^{n-1} + \frac{n(n-1)}{2}.(\delta x)^2.x^{n-2} + \cdots \end{aligned}$$

so $\frac{\delta y}{\delta x} = n.x^{n-1} + \delta x.\frac{n(n-1)}{2}x^{n-2} + \cdots$ and in the limit as δx tends to 0, all the terms on the right tend to 0 and we are left with $\frac{\mathrm{d}y}{\mathrm{d}x} = n.x^{n-1}$.

You can also differentiate $y = \sin x$ from first principles by Euclidean geometry, but if you don't like geometry you can use the trigonometric formula

$$\sin(x + y) = \sin x. \cos y + \cos x. \sin y$$

which implies that

$$\begin{aligned} y + \delta y &= \sin(x + \delta x) \\ &= \sin x. \cos \delta x + \cos x. \sin \delta x \\ &\approx \sin x + (\cos x).(\delta x) \\ &\quad \text{since } \sin \delta x \approx \delta x \text{ when } \delta x \text{ is small} \\ &= y + \delta x. \cos x \end{aligned}$$

so $\frac{\delta y}{\delta x} \approx \cos x$ and in the limit as δx tends to 0 we get $\frac{\mathrm{d}}{\mathrm{d}x}(\sin x) = \cos x$.

Differentiating a product If $y = uv$, where y, u and v are all functions of x, then suppose we increase the value of x to $x + \delta x$, with corresponding values of u, v and y being $u + \delta u$, $v + \delta v$, and $y + \delta y$. Then by definition

$$\begin{aligned} y + \delta y &= (u + \delta u)(v + \delta v) \\ &= uv + v.\delta u + u.\delta v + \delta u.\delta v \\ &= y + v.\delta u + u.\delta v + \delta u.\delta v \end{aligned}$$

Now cancelling out the y from both sides of the equation, and dividing by δx, gives us

$$\frac{\delta y}{\delta x} = v.\frac{\delta u}{\delta x} + u.\frac{\delta v}{\delta x} + \delta u.\frac{\delta v}{\delta x}$$

and in the limit as δx tends to 0 we get

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= v.\frac{\mathrm{d}u}{\mathrm{d}x} + u.\frac{\mathrm{d}v}{\mathrm{d}x} + 0.\frac{\mathrm{d}v}{\mathrm{d}x} \\ &= v.\frac{\mathrm{d}u}{\mathrm{d}x} + u.\frac{\mathrm{d}v}{\mathrm{d}x}\end{aligned}$$

Integration The next topic is *integration.* This has two quite different meanings, and it is far from obvious why the two are connected.

The first type of integration is *antidifferentiation*, that is just the opposite of differentiation, also called the *indefinite integral.*

The second type of integration is *the area under a curve*, also called the *definite integral.*

Antidifferentiation If $\frac{\mathrm{d}y}{\mathrm{d}x} = z$, say, i.e. you differentiate y to get z, then you integrate z to get back to y. We say that y is the *indefinite integral* (of z, with respect to x).

We write $y = \int z \,\mathrm{d}x$, and read "y is the integral of z with respect to x". The sign $\int$ is called the *integral sign*, and we use the symbol $\mathrm{d}x$ to show that the variable is x.

Example If $y = x$ then $\frac{\mathrm{d}y}{\mathrm{d}x} = 1$, so (using the above notation) $z = 1$, which means that $\int 1 \,\mathrm{d}x = x$.

But what happens if $y = x + 2$? Again we have $\frac{\mathrm{d}y}{\mathrm{d}x} = 1$, so by the above definition we have $\int 1 \,\mathrm{d}x = x + 2$.

Indeed, if we add *any constant* to y, we do not change $\frac{\mathrm{d}y}{\mathrm{d}x}$, so we cannot tell which constant we started with. Thus we normally write

$$\int 1 \,\mathrm{d}x = x + C,$$

with the C denoting an arbitrary constant. Usually, we then have to look at some other information in the problem, to determine what the correct value of C is in any given case.

Example If $y = x^2$ then $\frac{\mathrm{d}y}{\mathrm{d}x} = 2x$, so (using the above notation) $z = 2x$, which means that $\int 2x \,\mathrm{d}x = x^2 + C$. Dividing both sides by 2 gives

$$\int x \,\mathrm{d}x = \frac{x^2}{2} + k,$$

where k is again an arbitrary constant.

Example If $y = x^{n+1}$, where n is a positive integer, then

$$\frac{\mathrm{d}y}{\mathrm{d}x} = (n+1)x^n$$

so $z = (n+1)x^n$ and

$$\int (n+1)x^n \ \mathrm{d}x = x^{n+1} + C$$

and therefore

$$\int x^n \ \mathrm{d}x = \frac{x^{n+1}}{n+1} + k.$$

Example Actually, the above calculation works for n being a negative integer as well, and even for n being any real number, *except* for the case $n = -1$.

What goes wrong in this case?

The answer is that if $n = -1$ then $n + 1 = 0$, and in the last line of the calculation we have divided by 0. THIS IS NOT ALLOWED.

It is still true that

$$\int (n+1)x^n \ \mathrm{d}x = x^{n+1} + C$$

but now this only tells us that $\int 0 \ \mathrm{d}x = x^0 + C$, in other words $\int 0 \ \mathrm{d}x$ is a constant. But we knew this anyway, and it does not tell us anything about $\int x^{-1} \ \mathrm{d}x$.

In fact, since $\frac{\mathrm{d}}{\mathrm{d}x} \log_e(x) = \frac{1}{x}$, we have

$$\int \frac{1}{x} \ \mathrm{d}x = \log_e(x)$$

where $e \approx 2 \cdot 71828$. These logarithms are called *natural* logarithms, as they arise naturally in this integral. Thus the number e is called the *base of natural logarithms*, just as π is the "natural base" for the trigonometric functions.

When mathematicians write $\log(x)$, without specifying a base, they ALWAYS MEAN $\log_e(x)$. Many people write $\ln x$ for $\log_e(x)$.

Example We have $\frac{\mathrm{d}}{\mathrm{d}x}(\sin x) = \cos x$, so

$$\int \cos x \ \mathrm{d}x = \sin x,$$

and similarly

$$\int \sin x \ \mathrm{d}x = -\cos x.$$

Areas under curves If y is a function of x, and y is always positive, then you can ask for the area between the curve and the x-axis. For example, what is the area beneath the curve $y = \sin x$ and the x-axis, in the range $0 \leq x \leq \pi$? More generally, we define the *definite integral*

$$\int_a^b y \, \mathrm{d}x$$

to be the area under the curve between the values $x = a$ and $x = b$.

For example, if $y = c$, a constant, then the area in question is just a rectangle and

$$\int_a^b c \, \mathrm{d}x = c(b - a)$$

What happens if c is negative? Then this formula gives a negative value, and the area in question is below the x-axis. So be careful: areas below the x-axis come with a minus sign attached.

Take another example, such as $y = mx$. Then by calculating areas of triangles it is easy to see that

$$\int_a^b mx \, \mathrm{d}x = \frac{m}{2}b^2 - \frac{m}{2}a^2$$

which is the same as: (indefinite integral evaluated at $x = b$) − (indefinite integral evaluated at $x = a$). [Notice that the constants of integration cancel out.]

For example if $y = x^2$, so that $\int x^2 \, \mathrm{d}x = \dfrac{x^3}{3} + C$, then

$$\begin{aligned} \int_a^b x^2 \, \mathrm{d}x &= \left[\frac{x^3}{3} + C\right]_a^b \\ &= \left(\frac{b^3}{3} + C\right) - \left(\frac{a^3}{3} + C\right) \\ &= \frac{b^3}{3} - \frac{a^3}{3} \end{aligned}$$

More examples:

$$\begin{aligned} \int_0^\pi \sin x \, \mathrm{d}x &= [-\cos x]_0^\pi \\ &= -\cos \pi - (-\cos 0) \\ &= 2 \\ \int_1^a \frac{1}{x} \, \mathrm{d}x &= [\ln x]_1^a \\ &= \ln a - \ln 1 \\ &= \ln a \end{aligned}$$

Polar coordinates So far all our curves have been written as an equation involving x and y, which measure horizontal and vertical distance respectively. However, in practice many curves are easier to describe using *polar coordinates*, which means specifying the actual distance from a specified origin (the 'original point' you start from), and the direction. Here we specify the direction by the angle from the horizontal, measured *anticlockwise*. This angle is traditionally called θ, and the distance from the origin is called r.

Thus by Pythagoras, $r^2 = x^2 + y^2$, and trigonometry tells you that $\frac{y}{x} = \tan\theta$. So $r = \pm\sqrt{x^2 + y^2}$ and $\theta = \tan^{-1}\frac{y}{x}$. Have you spotted the deliberate mistake? What happens if x is negative? Well, in this case $\theta = \tan^{-1}\frac{y}{x} \pm \pi$. So there are two slight problems here: one is to determine the sign of r (common sense says it is always positive, but it also makes sense mathematically to consider negative values of r), and the other is to determine whether you need to add π onto the angle θ. You will need to use your common sense, and inspection of the diagram, to solve these problems.

This shows how to convert from (x, y)-coordinates to polar coordinates. What about going the other way? If we know r and θ, how do we calculate x and y? Well, $\sin\theta = \frac{y}{r}$, so $y = r\sin\theta$, and similarly $x = r\cos\theta$.

Example A circle of radius a centred at the origin has equation $r = a$, which is much simpler than the Cartesian version $x^2 + y^2 = a^2$.

Exercise Sketch the curve defined by the equation $r = \sin\theta$. Most people draw a rough egg-shape of height 1 balancing on the origin. But did you realise it was really a circle? To see this, translate it back into (x, y)-coordinates. Multiplying the equation through by r we get

$$\begin{aligned} r^2 &= r\sin\theta \\ \text{so } x^2 + y^2 &= y \\ x^2 + y^2 - y + \frac{1}{4} &= \frac{1}{4} \\ x^2 + (y - \frac{1}{2})^2 &= \frac{1}{4} \end{aligned}$$

which is the equation of a circle of radius $\frac{1}{2}$ centred at the point $(0, \frac{1}{2})$.

Exercise Sketch the curve $r = \theta$.

(Harder) Sketch the curve $r = \tan\theta$.