MSM120—1M1
First year mathematics for civil engineers
Revision notes 3

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Functions Definition of a function: it is a *rule* which, given a value of the *independent variable* (often but not always called x) determines the value of the *dependent variable* (often but not always called y). We write $y = f(x)$, to denote that y is a *function of* x, and the letter f here denotes the 'function' itself.

A simple example of a function is $f(x) = x^2$, which tells us that for any given value of x (such as $x = -2$), we get the corresponding value of y as x^2 (such as $y = x^2 = (-2)^2 = 4$).

The *domain* of a function is the set of allowed values of x. Usually our functions will be defined for any value of x, so the domain will be the set of all real numbers, which we write $\mathbb{R}$. Sometimes however we need to restrict to smaller sets. For example, $y = \dfrac{1}{x}$ is not defined for $x = 0$, so the domain in this case is the set of *all real numbers except 0.*

The *codomain* of a function is the set of allowed values of y (almost always $\mathbb{R}$ for us), not just the values of y which actually occur. The *range*, on the other hand, is just the set of values of y which actually occur. Example: $y = f(x) = x^2$, domain and codomain are both $\mathbb{R}$, range is $\mathbb{R}_{\geq 0}$.

Functions do not have to be given by a single formula. All you need is a *rule* that enables you to calculate $f(x)$, given any value of x.

Example The following is a perfectly good definition of a particular function f. Draw its graph.

$$y = f(x) \quad = \quad \begin{cases} 2 \text{ if } x \geq 2 \\ 1 \text{ otherwise} \end{cases}$$

Definition and examples of *sum* and *product* of two functions: if f and g are two functions, then $f+g$ is just the function you get by adding together the values of the two functions f and g. So for example if $f(x) = 2x^2$ and $g(x) = x - 1$, then

$$(f + g)(x) = 2x^2 + x - 1.$$

Similarly $f.g$ is the function you get by multiplying together the values of the two functions f and g. So in the above example

$$(f.g)(x) = 2x^2.(x-1) = 2x^3 - 2x^2.$$

Composition of functions If $y = f(x)$ and $z = g(y)$ you can substitute one into the other to get $z = g(f(x))$, so that z is a function of x. This function is written $g \circ f$, and is called the *composite function of g with f*.

Example If $f(x) = \sin x$ and $g(x) = 3x^2 - 1$, then by changing the name of the variable, we can write $g(y) = 3y^2 - 1$. Substituting in $y = f(x) = \sin x$ we get

$$(g \circ f)(x) = g(f(x)) = g(\sin x) = 3(\sin x)^2 - 1 = 3\sin^2 x.$$

On the other hand, if we compose the functions in the opposite order, by writing $f(y) = \sin y$ and $y = g(x)$ we get

$$(f \circ g)(x) = f(g(x)) = f(3x^2 - 1) = \sin(3x^2 - 1).$$

Notice that $f \circ g$ and $g \circ f$ are completely different functions.

Examples of functions *Polynomial* functions are functions of the form

$$f(x) = ax^n + bx^{n-1} + \cdots + k$$

where a, b, ..., k are constants. They are always defined for all values of x, that is, for all real numbers, so here $f : \mathbb{R} \to \mathbb{R}$. The positive integer n is called the *degree* of the polynomial, and $a, \ldots$ are the *coefficients*.

Rational functions are functions of the form one polynomial divided by another. These are in general not defined for all real values of x. For example, $f(x) = \frac{1}{x}$ is not defined when $x = 0$. More generally, a rational function $f(x)/g(x)$, where $f(x)$ and $g(x)$ are polynomials, will not be defined when $g(x) = 0$.

A rational function of this form is called *proper* if the degree of f is smaller than the degree of g. Otherwise, it is called *improper*, and we can divide $g(x)$ into $f(x)$ (using long division of polynomials) to get a quotient $q(x)$ and a remainder $r(x)$. This means that

$$\frac{f(x)}{g(x)} = q(x) + \frac{r(x)}{g(x)}$$

and so the improper rational function has been expressed as the sum of a polynomial and a proper rational function.

Try this yourself with an example:

$$\frac{3x^3 - x + 1}{x^2 + 2}$$

A proper rational function may then be simplified further using the method of *partial fractions*.

Partial fractions This method is also very important for integrating rational functions. If you are given a complicated rational function to integrate, then you will have no hope unless you simplify it as much as possible first.

For example, a proper rational function such as $\dfrac{x+2}{(x-1)(x+1)}$ can be written in the form $\dfrac{A}{x-1}+\dfrac{B}{x+1}$, where A and B are constants—which I am sure you will agree is much simpler. To find the values of A and B, first clear denominators:

$$\begin{aligned}\frac{x+2}{(x-1)(x+1)} &= \frac{A}{x-1}+\frac{B}{x+1}\\ x+2 &= A(x+1)+B(x-1)\end{aligned}$$

Now the important point to note is that this equation is supposed to be true for *all* values of x, so we can substitute in helpful values of x if we like. For example, if $x=1$, then the term involving B disappears, and we get $3 = A.2$, so $A=\frac{3}{2}$. Similarly, if $x=-1$, then the term involving A disappears, giving $1 = B.(-2)$, so $B=\frac{-1}{2}$.

The same method works for any number of linear factors in the denominator, provided there are no repeated factors. If there are repeated factors, use the following method: as an example, take $\dfrac{x^2+x}{(x-1)^3}$. This time we can reduce it to the form

$$\begin{aligned}\frac{x^2+x}{(x-1)^3} &= \frac{A}{x-1}+\frac{B}{(x-1)^2}+\frac{C}{(x-1)^3}\\ \text{so } x^2+x &= A(x-1)^2+B(x-1)+C\end{aligned}$$

Now putting $x=1$ gives $2=C$. We can also equate coefficients—because we have an equality of polynomials here, which means that the two polynomials are equal for *all* values of x, all the coefficients must be the same. So the coefficient of x^2 on the left hand side is 1, while the coeffecient of x^2 on the right hand side is A, so $A=1$. Similarly, the constant term on the left hand side is 0, while on the right hand side it is $A-B+C$, so $A-B+C=0$, but we already know $A=1$ and $C=2$, so $B=3$.

If you have quadratic factors which cannot be factorised into linear factors, then you need something of the form $Ax+B$ on top of them. For example,

$$\begin{aligned}\frac{x^2-2}{(x-1)(x^2+1)} &= \frac{A}{x-1}+\frac{Bx+C}{x^2+1}\\ x^2-2 &= A(x^2+1)+(Bx+C)(x-1)\\ &= Ax^2+A+Bx^2-Bx+Cx-C\\ &= (A+B)x^2+(C-B)x+(A-C)\end{aligned}$$

Putting $x = 1$ in the second line gives $2 = 2A$ so $A = 1$. Now equating coefficients of x gives $B = C$, and equating constant terms gives $-2 = A - C = 1 - C$ so $C = 3$, and therefore $B = 3$.

Sketching the graph of a rational function First simplify it as above. The polynomial part tells you approximately what the function looks like as x tends to $\pm\infty$, because when x is very large the proper rational functions are very small. The linear factors in the denominator tell you the points where the function is not defined—near to these points the function shoots off to either $+\infty$ or $-\infty$ or both. Plot some points to tell you which.

Inverse functions If we are given y as a function of x, we often want to invert the relationship, and express x as a function of y instead. For example, if $y = 3x - 2$ then we deduce $y + 2 = 3x$ and so $x = \frac{1}{3}(y + 2)$. In functional notation, we started with $y = f(x)$, where $f(x) = 3x - 2$, and ended up with $x = g(y)$, where $g(y) = \frac{1}{3}(y + 2)$, or, by changing the name of the variable but not changing the function g, $g(x) = \frac{1}{3}(x + 2)$.

Consider another example: $y = x^2$. Then we have $x = \pm\sqrt{y}$ so how do we know which sign to take? In general you don't, so this function DOES NOT HAVE AN INVERSE. However, if you restrict to the case when $x \geq 0$, then you know $x = +\sqrt{y}$, and everything works. So this requires us to restrict the DOMAIN of the original function to $\mathbb{R}_{\geq 0}$.

Similarly with a function like $y = \sin x$. The general solution of this is $x = \pm(\frac{\pi}{2} + \sin^{-1} y) + (2n - \frac{1}{2})\pi$, so to get a unique solution we need to restrict to a suitable domain for x, such as $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$.

Hyperbolic functions The function e^x arises frequently in engineering applications, for example in solutions of differential equations, but more often than not there is a symmetry involved, so that the function that actually arises is $e^x + e^{-x}$ or $e^x - e^{-x}$. These functions (divided by 2 to make things easier later on) are therefore given special names.

$$\begin{aligned}
\cosh x &= \frac{e^x + e^{-x}}{2} \\
\sinh x &= \frac{e^x - e^{-x}}{2} \\
\tanh x &= \frac{\sinh x}{\cosh x} \\
\operatorname{sech} x &= \frac{1}{\cosh x} \\
\operatorname{cosech} x &= \frac{1}{\sinh x}
\end{aligned}$$

$$\coth x = \frac{1}{\tanh x}$$

Exercise Sketch the graphs of these functions.

There are many identities between hyperbolic functions, analogous to the trigonometric identities which you should have seen already.

For example, corresponding to $\cos^2 x+\sin^2 x = 1$ we have $\cosh^2 x-\sinh^2 x = 1$. We can deduce this easily from the definitions, as follows:

$$\begin{aligned}
\cosh^2 x - \sinh^2 x &= \left(\frac{e^x + e^{-x}}{2}\right)^2 - \left(\frac{e^x - e^{-x}}{2}\right)^2 \\
&= \left(\frac{e^x}{2}\right)^2 + 2\frac{e^x}{2}\frac{e^{-x}}{2} + \left(\frac{e^{-x}}{2}\right)^2 \\
&\quad - \left\{\left(\frac{e^x}{2}\right)^2 - 2\frac{e^x}{2}\frac{e^{-x}}{2} + \left(\frac{e^{-x}}{2}\right)^2\right\} \\
&= \frac{4e^0}{4} = 1
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
\cosh(2t) &= 2\cosh^2 t - 1 \\
\sinh(x+y) &= \sinh x \cosh y + \cosh x \sinh y
\end{aligned}$$

etc. etc. Many of these identities can be found on your formula sheet.

Since you know that the derivative of e^x is e^x, you can easily work out the derivatives of the hyperbolic functions. For example

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(\cosh x) &= \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{e^x + e^{-x}}{2}\right) \\
&= \frac{e^x - e^{-x}}{2} \\
&= \sinh x
\end{aligned}$$

and similarly

$$\frac{\mathrm{d}}{\mathrm{d}x}(\sinh x) = \cosh x.$$

Also

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(\tanh x) &= \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\sinh x}{\cosh x}\right) \\
&= \frac{\cosh x \cosh x - \sinh x \sinh x}{\cosh^2 x} \\
&= \frac{1}{\cosh^2 x}
\end{aligned}$$

Equations with hyperbolic functions in them can often be solved by substituting in the definitions (in terms of e^x), and multiplying up by a common denominator. For example, to solve

$$5\cosh x + 3\sinh x = 4$$

we write it in the following equivalent forms:

$$\begin{aligned} \frac{5}{2}(e^x + e^{-x}) + \frac{3}{2}(e^x - e^{-x}) &= 4 \\ 4e^x + e^{-x} &= 4 \\ 4e^{2x} - 4e^x + 1 &= 0 \\ (2e^x - 1)^2 &= 0 \end{aligned}$$

from which we deduce that $e^x = \frac{1}{2}$, so $x = \ln\frac{1}{2} = -\ln 2$.

Inverse hyperbolic functions If $y = \sinh x$ then by the definition of inverse functions, we have $x = \sinh^{-1} y$. The other inverse hyperbolic functions are definied similarly. In fact, they all have alternative expressions in terms of logarithms, which can be deduced by the same method we have just used for solving equations.

Suppose $y = \sinh x = \frac{1}{2}(e^x - e^{-x})$, and we want to solve for x as above. Then multiply up by $2e^x$ to get $2.e^x.y = e^{2x} - 1$, which we rewrite as a quadratic equation in e^x

$$(e^x)^2 - 2y(e^x) - 1 = 0$$

so $e^x = y \pm \sqrt{y^2+1}$. Now if we took the negative square root, then the expression $y - \sqrt{y^2+1}$ would be negative, whereas e^x is always positive, so this case cannot happen. Therefore we have to take the positive square root, and taking logs then gives

$$x = \sinh^{-1} y = \ln(y + \sqrt{y^2+1}).$$

Similarly we can obtain the identity

$$\cosh^{-1} y = \ln(y + \sqrt{y^2-1}).$$

But notice in this case that there are two possible values of x corresponding to each possible value of $\cosh x$, since $\cosh(-x) = \cosh x$. By convention $\cosh^{-1} y$ is taken to be the positive value of x.

Parametric functions Often a curve is difficult to describe just with an equation in x and y coordinates (or polar coordinates r and θ) and may be easier to describe by having both x and y as functions of a third variable, traditionally called t.

For example, a circle of radius a centred at the origin can be described by the equations

$$\begin{aligned} x &= a\cos t \\ y &= a\sin t \end{aligned}$$

Similarly, one half of the rectangular hyperbola $x^2 - y^2 = a^2$ can be described by

$$\begin{aligned} x &= a\cosh t \\ y &= a\sinh t \end{aligned}$$

(This explains why these functions are called *hyperbolic functions*.)

Exercise Sketch the curve defined by the equations

$$\begin{aligned} x &= t^2 \\ y &= t^3 - 2t \end{aligned}$$

Applications of hyperbolic functions A rope or cable hanging under its own weight hangs in the shape of a hyperbolic cosine. If you take a suitable origin of coordinates, then the equation is of the form $y = A\cosh(Bx)$, where A and B are constants depending on the dimensions.

Now consider a suspension bridge. First you build the pylons to support the weight. Then you hang a heavy cable between the tops of the pylons—this has to be heavy because it supports the entire weight of the bridge. It hangs in the shape of $y = A\cosh(Bx)$. Then you hang light vertical cables from the main cable, each supporting a small part of the deck. Since these are light, they do not alter the shape of the main cable much. The deck itself is flat, so applies an evenly balanced load across the width of the bridge. So this doesn't affect the shape of the main cable either.

Another application concerns two rivers, travelling at different speeds, meeting. After the rivers have joined together, the speed varies across the river according to a function of the form $y = A\tanh(Bx) + C$.