Queen Mary University of London

MTHM024/MTH714U Group Theory

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9 Soluble and nilpotent groups

9.1 Soluble groups

There are several ways to recognise when a finite group is soluble. Recall that the *derived group* or *commutator subgroup* G' of G is the subgroup generated by all *commutators* $[g,h] = g^{-1}h^{-1}gh$ for $g,h \in G$. It is a normal subgroup of G with the properties that G/G' is abelian, and if N is any normal subgroup of G such that G/N is abelian, then $G' \leq N$. Inductively we define $G^{(r)}$ for $r \in \mathbb{N}$ by $G^{(0)} = G$ and $G^{(r+1)} = (G^{(r)})'$ for $r \geq 0$.

Note that, if $G^{(i)} = G^{(i+1)}$, then $G^{(i)} = G^{(j)}$ for all $j > i$.

Theorem 9.1 *For the finite group G, the following properties are equivalent:*

(a) There is a chain of subgroups

$$G = G_0 \geq G_1 \geq G_2 \geq \cdots \geq G_{r-1} \geq G_r = \{1\}$$

such that $G_i \lhd G_{i-1}$ and G_{i-1}/G_i is cyclic of prime order for $i = 1,2,\ldots,r$ (in other words, all the composition factors of G are cyclic of prime order);

(b) There is a chain of subgroups

$$G = H_0 \geq H_1 \geq H_2 \geq \cdots \geq H_{s-1} \geq H_s = \{1\}$$

such that $H_i \lhd G$ and H_{i-1}/H_i is abelian for $i = 1,2,\ldots,s$;

(c) there exists r such that $G^{(r)} = \{1\}$.

Proof (c) implies (b): If $G^{(r)} = \{1\}$, then the subgroups $H_i = G^{(i)}$ satisfy the conditions of (b).

(b) implies (a): Suppose that we have a chain of subgroups as in (b). Now if A is a finite abelian group, then A has a composition series with cyclic composition factors of prime order. (The proof is by induction. Working from the bottom up, let $H_s = \{1\}$ and H_{s-1} the subgroup generated by an element of prime order; using the inductive property, choose a composition series for A/H_{s-1}, and use the Correspondence Theorem to lift them to a composition series of A containing H_{s-1}.)

Now choose a composition series in each abelian quotient, and lift each to a part of a composition series between G_{i-1} and G_i.

(a) implies (c): We use the fact that, if G/N is abelian, then $G' \leq N$. If a composition series with prime cyclic factor groups exists as in (a), then by an easy induction, the ith term $G^{(i)}$ in the derived series is contained in G_i; so $G^{(r)} = \{1\}$.

The *derived length* or *soluble length* of the soluble group G is the minimum r such that $G^{(r)} = \{1\}$. Note that a non-trivial finite group is abelian if and only if it is soluble with derived length 1.

Theorem 9.2 *(a) Subgroups, quotient groups, and direct products of soluble groups are soluble.*

(b) If G has a normal subgroup N such that N and G/N are soluble, then G is soluble.

Proof (a) If $H \leq G$ then all commutators of elements of H belong to G', and so $H' \leq G'$. By induction, $H^{(i)} \leq G^{(i)}$ for all i. So, if $G^{(r)} = \{1\}$, then $H^{(r)} = \{1\}$.

If $N \leq G$, then $[Ng, Nh] = N[g,h]$, so $(G/N)' = G'N/N$. By induction, $(G/N)^{(i)} = G^{(i)}N/N$ for all i. So, if $G^{(r)} = \{1\}$, then $(G/N)^{(r)} = \{1\}$.

In $G \times H$, we have $[(g_1,h_1),(g_2,h_2)] = ([g_1,g_2],[h_1,h_2])$ for all $g_1, g_2 \in G$ and $h_1, h_2 \in H$. So $(G \times H)' = G' \times H'$. By induction, $(G \times H)^{(i)} = G^{(i)} \times H^{(i)}$ for all i. So, if $G^{(r)} = \{1\}$ and $H^{(s)} = \{1\}$, then $(G \times H)^{(t)} = \{1\}$, where $t = \max\{r,s\}$.

Suppose that $N^{(r)} = \{1\}$ and $(G/N)^{(s)} = \{1\}$. Arguing as in (a), we see that $G^{(s)} \leq N$, and so $G^{(r+s)} = \{1\}$.

Remark The arguments in the proof show that the derived length of a subgroup or quotient of G are not greater than the derived length of G, while the derived length of a direct product is the maximum of the derived length of the factors.

9.2 Nilpotent groups

Recall that the *centre* of G is the subgroup $Z(G) = \{g \in G : gx = xg \text{ for all } x \in G\}$. It is an abelian normal subgroup of G. Now we define a series of subgroups of G called the *upper central series* of G as follows: $Z_0(G) = \{1\}$, $Z_{i+1}(G)$ is the normal subgroup of G corresponding to the centre of $G/Z_i(G)$ by the Correspondence Theorem. (Briefly we say $Z_{i+1}(G)/Z_i(G) = Z(G/Z_i(G))$.)

Note that, if $Z_i(G) = Z_{i+1}(G)$ (that is, if the centre of $G/Z_i(G)$ is trivial), then $Z_i(G) = Z_j(G)$ for all $j > i$.

The group G is said to be *nilpotent* if $Z_r(G) = G$ for some r; the smallest such r is called the *nilpotency class* of G. Again, a non-trivial finite group is abelian if and only if it is nilpotent with nilpotency class 1.

Theorem 9.3 *The following conditions for a finite group G are equivalent:*

(a) $Z_r(G) = G$ *for some* r*;*

(b) there is a chain of subgroups

$$G = H_0 \geq H_1 \geq H_2 \geq \cdots \geq H_{s-1} \geq H_s = \{1\}$$

such that $H_i \lhd G$ *and* $H_{i-1}/H_i \leq Z(G/H_i)$ *for* $i = 1, 2, \ldots, s$*;*

(c) all Sylow subgroups of G are normal;

(d) G is the direct product of its Sylow subgroups.

Thus nilpotency of a finite group can be defined by any of the equivalent conditions of the Theorem. (As for solubility, the conditions are no longer equivalent for infinite groups.) Note that

(a) a nilpotent group is soluble (for the centre of a group is abelian, so the quotients of the groups in the chain (b) are abelian);

(b) a group of prime power order is nilpotent;

(c) the smallest non-abelian group, S_3, is soluble but not nilpotent.

Proof (a) implies (b): If $Z_r(G) = G$, then the subgroups $H_i = Z_{r-i}(G)$ satisfy the conditions of (b).

(b) implies (c): We defer this for a moment.

(c) implies (d): This is proved by a straightforward induction on the number of primes dividing $|G|$.

(d) implies (a): Recall that, if P is a non-trivial group of prime-power order, then $Z(P) \neq \{1\}$. Thus, by induction, a group of prime-power order satisfies (a). Moreover, it is easy to see that

$$Z(P_1 \times \cdots \times P_m) = Z(P_1) \times \cdots \times Z(P_m);$$

so a direct product of groups satisfying (a) also satisfies (a).

The remaining implication is a little more difficult; it follows from a couple of lemmas which we now prove.

Lemma 9.4 *Let P be a Sylow p-subgroup of the group G, and H a subgroup of G which contains the normaliser $N_G(P)$ of P. Then $N_G(H) = H$.*

Proof Take $g \in N_G(H)$, so that $g^{-1}Hg = H$. Then $g^{-1}Pg \leq H$, so $g^{-1}Pg$ is a Sylow p-subgroup of H. By Sylow's theorem, all the Sylow p-subgroups of H are conjugate, so there exists $h \in H$ satisfying $h^{-1}(g^{-1}Pg)h = P$. Then $gh \in N_G(P) \leq H$, so $gh \in H$. Since $h \in H$, it follows that $g \in H$. So $N_G(H) = H$, as claimed.

Remark This argument is known as the *Frattini argument*.

Lemma 9.5 *Suppose that G satisfies (b) of the Theorem. If H is a proper subgroup of G, then $H < N_G(H)$.*

Proof Let i be maximal such that $G_i \leq H$. Then $i \neq 0$, since $H < G$. Now $G_{i-1} \not\leq H$, so there is a coset $G_i g$ in G_{i-1}/G_i which is not in H/G_i but commutes with all cosets of G_i, and hence normalises H/G_i. Thus, $N_{G/G_i}(H/G_i) > H/G_i$. Judicious use of the Correspondence Theorem shows that $N_G(H) > H$.

Now we can show that (b) implies (c) in the theorem. Suppose that G satisfies (b), and let P be a Sylow p-subgroup of G, for some prime p. Let $H = N_G(P)$. Then $N_G(H) = H$. But if $H < G$, then $N_G(H) > H$; so we must have $H = G$ as required.

Remark The condition

If $H < G$, then $H < N_G(H)$

is equivalent to the four conditions of the theorem, and so provides another equivalent to nilpotency of a finite group. [Can you prove this?]

Exercise Prove that subgroups, quotients and direct products of nilpotent groups are nilpotent.

9.3 Supersoluble groups

A finite group G is *supersoluble* if there is a chain

$$G = G_0 > G_1 > G_2 > \cdots > G_{r-1} > G_r = \{1\}$$

of subgroups such that $G_i \rhd G$ and G_{i-1}/G_i is cyclic of prime order for $i = 1,2,\ldots,r$.

Look back at the first theorem of this chapter. In a soluble group G, we may assume *either* that all the subgroups in the chain are normal in G (with the quotients being abelian), *or* that all the quotients are cyclic of prime order (with each subgroup being normal in the one before). The example $G = A_4$ shows that we cannot ask both things in general. The only candidate for G_1 is V_4, which is not cyclic; its cyclic subgroups of order 2 are not normal in G. In other words, A_4 is not supersoluble. However, S_3 is supersoluble.

Supersoluble groups form a class between nilpotent and soluble. (Any nilpotent group is supersoluble, because a subgroup contained in the centre of a group G is normal.) They are not as important as either nilpotent or soluble groups. Here is a surprising fact about them.

Theorem 9.6 *If G is supersoluble, then G' is nilpotent.*

Proof This depends on the fact that the automorphism group of a cyclic group of prime order is abelian. (In fact, $\mathrm{Aut}(C_p) = C_{p-1}$.) Hence, a homomorphism from G to $\mathrm{Aut}(C_p)$ has the property that its kernel contains G'.

Let

$$G = G_0 > G_1 > G_2 > \cdots > G_{r-1} > G_r = \{1\}$$

be a series of subgroups such that $G_i \lhd G$ and G_{i-1}/G_i is cyclic of prime order for $i = 1,2,\ldots,r$. Now consider the series

$$G' = H_0 \geq H_1 \geq H_2 \geq \cdots \geq H_{r-1} \geq H_r = \{1\},$$

where $H_i = G_i \cap G'$. Then $H_i \lhd G'$, and

$$H_{i-1}/H_i = (G_{i-1} \cap G')/(G_i \cap (G_{i-1} \cap G')) \cong (G_{i-1} \cap G')G_i/G_i \leq G_{i-1}/G_i,$$

so H_{i-1}/H_i is either trivial or cyclic of prime order. By dropping terms from the series, we can assume it is always cyclic of prime order.

Now G acts by conjugation on H_{i-1}/H_i. By our earlier remark, G' acts trivially on this quotient, which means that $H_{i-1}/H_i \leq Z(G'/H_i)$. Since this is true for all i, we see that G' is nilpotent.