An Atlas of Sporadic Group Representations

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Introduction

The 'ATLAS of Finite Groups' [5] was originally conceived by its authors as Volume 1 of a series, as its subtitle 'Maximal subgroups and ordinary characters for simple groups' might suggest. In the event, subsequent volumes have been rather slow to appear, with Volume 2, the 'Atlas of Brauer Characters' (or 'ABC' for short [8]), being published in 1995, just in time for this conference. Indeed, even this is only Part 1 of Volume 2, as the accidentally undeleted subtitle on page 1 proclaims, in that it only includes groups of order up to 10^9.

At this conference, several suggestions for Volume 3 have been made, most involving large quantities of data stored on computers. It seems likely that whatever Volume 3 eventually turns out to be, it will not be a big heavy book of the type hitherto associated with the word 'Atlas'.

My own submission as a candidate for Volume 3 is a collection of explicit representations of groups. A number of these were mentioned in the 'Atlas of Finite Groups' under the now notorious phrase 'Explicit matrices have been computed.' Many others have been computed since. In fact, it is difficult to know where to stop with such a collection of representations, and it (like many databases) could easily be allowed to expand to fill all the disk-space available.

For the moment, this 'Atlas of group representations' is restricted to the sporadic simple groups, and we now have (in principle) representations of all of these and their covers and automorphism groups, with the two exceptions of the Monster and the double cover of the Baby Monster. A list of the groups and representations that are included is given in Table 1, though it will surely be out-of-date by the time it is printed. The reader may object that there is no representation of $6{}^{\cdot}Fi_{22}$ or $6{}^{\cdot}Fi_{22}{:}2$ in the list, but this should not matter too much, as representations of $2{}^{\cdot}Fi_{22}{:}2$ and $3{}^{\cdot}Fi_{22}{:}2$ are given. (In fact, there seems to be no convenient faithful representation of $6{}^{\cdot}Fi_{22}{:}2$ to construct. Perhaps the best would be as a group of permutations on 370656 points.)

One obvious direction in which this 'Atlas' could be extended is to include all exceptional covers of generic groups. There seems to be no serious obstacle (except lack of time and energy) to doing so, although it may be quite a challenge to construct a representation of $(2^2 \times 3)^{\cdot 2}E_6(2){:}S_3$. Another possible direction is to consider characteristic 0 matrix representations of reasonably small degree—this is an obvious area for application of R. A. Parker's new Integral Meat-axe [23].

The representations collected here have been obtained in various ways, which can be roughly divided into the following four categories.

1. From existing literature on hand constructions, or constructions involving limited use of a computer. For example, some representations of the Mathieu groups and Leech lattice groups, as well as J_1 and Ru.

2. From existing computer constructions. For example, representations of $3{}^{\cdot}J_3$, $3{}^{\cdot}McL$, the Fischer groups, Ly, Th, and J_4.

3. By constructing representations *ab initio*. For example, representations of $4{}^{\cdot}M_{22}$, HN, B, and new representations of $O'N$, Ly, Fi_{24} and He.

4. By constructing new representations from old ones. For example, other representations of the Mathieu groups, J_1, Ru, Suz, Fi_{22}, and others.

We now consider these various methods in more detail.

1 Existing 'hand' constructions

Here we also include some constructions which were originally computer-assisted, but which are small enough for group generators to be entered by hand. Most of these constructions are derived either from the constructions of M_{12} and M_{24} by Mathieu [16], [17], or the construction of Co_1 by Conway [2]. In particular, all the listed permutation representations of the Mathieu groups are easily obtained in this way. Similarly, the 24-dimensional characteristic zero matrix representation of $2{}^{\cdot}Co_1$ can be written over the integers and reduced modulo any prime, and easily gives rise to all the listed matrix representations of Co_2, Co_3, Suz and its decorations, as well as those of dimension 20–22 for HS and McL.

Other constructions of this type are the 28-dimensional representation of $2{}^{\cdot}Ru$ by Conway and Wales [4] (see also Conway's later simplification [1]), as well as the 36-dimensional representation of $3{}^{\cdot}J_3{:}2$ [3]. Easiest of all is the 7-dimensional representation of J_1 described by Janko [7]. Perhaps here

Table 1: Available representations of sporadic groups

Group	Degree	Field
M_{11}	11	1
	12	1
	10	2
	5	3
	16	4
M_{12}	16	4
$M_{12}{:}2$	24	1
	10	2
$2{\cdot}M_{12}$	24	1
$2{\cdot}M_{12}{:}2$	48	1
	10	3
	12	3
J_1	266	1
	1045	1
	1463	1
	1540	1
	1596	1
	2926	1
	4180	1
	20	2
	56	9
	56	5
	31	7
	7	11
	22	19
M_{22}	22	1
$M_{22}{:}2$	22	1
	77	1
	10	2
$2{\cdot}M_{22}$	10	7
$2{\cdot}M_{22}{:}2$	10	7
$3{\cdot}M_{22}$	6	4
	21	25
	21	7
	21	121
$3{\cdot}M_{22}{:}2$	12	2

Group	Degree	Field
$4{\cdot}M_{22}$	56	25
	16	49
	56	121
$4{\cdot}M_{22}{:}2$	32	7
$6{\cdot}M_{22}$	36	121
$6{\cdot}M_{22}{:}2$	72	11
$12{\cdot}M_{22}$	48	25
	24	121
$12{\cdot}M_{22}{:}2$	48	11
J_2	6	4
$J_2{:}2$	100	1
	14	5
$2{\cdot}J_2$	6	9
$2{\cdot}J_2{:}2$	12	3
M_{23}	23	1
	11	2
	280	23
HS	100	1
	176	1
$HS{:}2$	100	1
	352	1
	15400	1
	20	2
	22	3
	21	5
	896	11
$2{\cdot}HS$	28	5
$2{\cdot}HS{:}2$	112	3
	56	5
J_3	18	9
$J_3{:}2$	6156	1
	36	3
$3{\cdot}J_3$	9	4
$3{\cdot}J_3{:}2$	18	2
M_{24}	24	1

Group	Degree	Field	Group	Degree	Field
McL	21	5	Co_2	2300	1
	1200	25		4600	1
McL:2	21	5		22	2
3·McL	396	4		24	2
	45	25		23	3
3·McL:2	90	5	Fi_{22}	3510	1
He	51	4	Fi_{22}:2	3510	1
	51	25		78	2
He:2	2058	1		77	3
	102	2	2·Fi_{22}	28160	1
	102	5		176	3
	50	7		352	5
Ru	4060	1	2·Fi_{22}:2	352	3
	28	2	2·Fi_{22}·4	352	5
2·Ru	16240	1	3·Fi_{22}	27	4
	28	5	3·Fi_{22}:2	54	2
Suz	1782	1	HN	132	4
Suz:2	64	3	HN:2	264	2
	1782	1		133	5
2·Suz	12	3	Ly	111	5
2·Suz:2	12	3		651	3
3·Suz	12	4		2480	4
3·Suz:2	24	2	Th	248	2
	5346	1		248	3
6·Suz	24	3	Fi_{23}	31671	1
6·Suz:2	24	3		782	2
$O'N$	154	3		253	3
$O'N$	406	7	Co_1	24	2
$O'N$:2	154	9	2·Co_1	24	3
$O'N$:4	154	3	J_4	112	2
3·$O'N$	153	4	Fi'_{24}:2	306936	1
3·$O'N$:2	90	7		781	3
	306	2	3·Fi'_{24}	920808	1
Co_3	22	2		783	4
	22	3	3·Fi'_{24}:2	1566	2
	276	1	B	4370	2

Note: we adopt the convention that an underlying ‘field’ of order 1 signifies a permutation representation.

we should also mention the 9-dimensional representation of $3{}^{\cdot}J_3$ over $GF(4)$, first found by Richard Parker using the Meat-axe, later tidied up by Benson and Conway into the form given in the ATLAS.

Some small permutation representations which can be obtained by hand include the representations of HS and J_2, and their automorphism groups, on 100 points, and $HS{:}2$ on 352 points.

2 Existing computer constructions

Under this heading there are some famous constructions, as well as a number of unpublished ones which have often been duplicated. In addition, there are some constructions which were essentially done by hand, but which are simply too big to enter into a computer in any simple-minded manner.

Perhaps the most important original matrix construction was that of J_4 in dimension 112 over $GF(2)$ by Norton and Parker [20], pioneering a technique that has since become standard (see [24]). Other important ones to mention are Parker's construction [18] of the Lyons group in $O_{111}(5)$, and $3{}^{\cdot}O'N$ in $GL_{45}(7)$ (see [27]), as well as $3{}^{\cdot}Fi_{22}$ in $GU_{27}(2)$. The Thompson group was constructed by Smith [30] in characteristic 0, and an analogous construction in characteristic 3 was given by Linton [12]. An explicit construction of the Held group was given by Ryba [26].

There are also some important constructions of permutation representations by coset enumeration. An early example was the construction of J_3 by Higman and McKay as a permutation group on 6156 points [6]. The permutation representations of the Fischer groups can also be considered in this category, especially the representations of Fi_{24} and $3{}^{\cdot}Fi_{24}$ which were provided for us by Steve Linton, using his double-coset enumerator [13].

3 *Ab initio* constructions

The impetus to start making a systematic collection of matrix representations of the sporadic groups came in June 1991 when Klaus Lux asked me for representations of several of the large sporadic groups. Searching through my files, I found two or three of these, but did not find HN or Fi_{24}. Accordingly, I tried to construct HN, choosing the 133-dimensional orthogonal representation over $GF(5)$. This construction took me three days [29], so I next tackled the 781-dimensional orthognal representation of Fi_{24} over $GF(3)$, which took only two days [36]. The following week I constructed the 4370-dimensional representation of the Baby Monster over $GF(2)$ (see [35]). In each case, the construction follows Parker's method [24].

Subsequently I returned to the subject at intervals when suitable interesting construction problems presented themselves. Ibrahim Suleiman and I gave the first (so far as we are aware) explicit construction of $4{}^{\cdot}M_{22}$, a group which a few years previously had been 'proved' not to exist.

While working with Christoph Jansen on computing modular character tables, we decided to tackle the very challenging problem of determining the 2-modular character table of the O'Nan group. After some exploratory calculations Christoph suggested that it was possible that the reduction of the degree 495 characters modulo 2 might contain a degree 342 character—if so then $3{}^{\cdot}O'N$ would have an irreducible (unitary) 153-dimensional representation over $GF(4)$. It seemed clear to me that the easiest way to prove this would be to construct the representation from scratch—which we did the next day [9]. (In fact, we went on to complete the 2-modular character table [10] soon afterwards.)

By a remarkable coincidence, a very similar chain of events led us to the construction of a 154-dimensional orthogonal representation of $O'N$ over $GF(3)$. This suggested to us that we should look for other such 'surprising' representations to construct, and in fact we showed that there was only one more (with a suitable definition of 'surprising'), namely a 651-dimensional orthogonal representation of the Lyons group over $GF(3)$, a construction of which is described in [11]. A similar construction gives the 2480-dimensional unitary representation over $GF(4)$ (see [37]).

4 Standard generators

Before considering the various ways of constructing new representations from old ones, it is worth pausing briefly to discuss generators for the groups. Each representation is most conveniently stored as a list of matrices (or permutations) giving the images of certain group generators in that representation. For various reasons it is important to standardize the generators for each given group. For example, the tensor product of two representations of a group G can only be made if the same generators for G are available in both representations. A discussion of some of the issues involved in choosing such 'standard generators' can be found in [38], and some implications and applications are explored in [33], using the specific example of the group J_3.

Here we simply list in Tables 2 and 3 the defining properties of our standard generators for the groups G and $G{:}2$, where G is a sporadic simple group. For the present, we consider generators for covering groups to be standard if they map to standard generators of G or $G{:}2$ under the natural quotient map. Thus they are *not* (yet) defined up to automorphisms.

Table 2: Standard generators of sporadic simple groups

Group	Triple (a, b, ab)	Further conditions
M_{11}	$2, 4, 11$	$o((ab)^2(abab^2)^2ab^2) = 4$
M_{12}	$2B, 3B, 11$	none
J_1	$2, 3, 7$	$o(abab^2) = 19$
M_{22}	$2A, 4A, 11$	$o(abab^2) = 11(\iff o(ab^2) = 5)$
J_2	$2B, 3B, 7$	$o([a, b]) = 12$
M_{23}	$2, 4, 23$	$o((ab)^2(abab^2)^2ab^2) = 8$
HS	$2A, 5A, 11$	none
J_3	$2A, 3A, 19$	$o([a, b]) = 9$
M_{24}	$2B, 3A, 23$	$o(ab(abab^2)^2ab^2) = 4$
McL	$2A, 5A, 11$	$o((ab)^2(abab^2)^2ab^2) = 7$
He	$2A, 7C, 17$	none
Ru	$2B, 4A, 13$	none
Suz	$2B, 3B, 13$	$o([a, b]) = 15$
$O'N$	$2A, 4A, 11$	none
Co_3	$3A, 4A, 14$	none
Co_2	$2A, 5A, 28$	none
Fi_{22}	$2A, 13, 11$	$o((ab)^2(abab^2)^2ab^2) = 12$
HN	$2A, 3B, 22$	$o([a, b]) = 5$
Ly	$2, 5A, 14$	$o(ababab^2) = 67$
Th	$2, 3A, 19$	none
Fi_{23}	$2B, 3D, 28$	none
Co_1	$2B, 3C, 40$	none
J_4	$2A, 4A, 37$	$o(abab^2) = 10$
${Fi_{24}}'$	$2A, 3E, 29$	$o((ab)^3b) = 33$
B	$2C, 3A, 55$	$o((ab)^2(abab^2)^2ab^2) = 23$
M	$2A, 3B, 29$	none

Table 3: Standard generators of automorphism groups of sporadic groups

Group	Triple (a, b, ab)	Further conditions
$M_{12}{:}2$	$2C, 3A, 12$	$ab \in 12A(\iff o([a, b]) = 11)$
$M_{22}{:}2$	$2B, 4C, 11$	none
$J_2{:}2$	$2C, 5AB, 14$	none
$HS{:}2$	$2C, 5C, 30$	none
$J_3{:}2$	$2B, 3A, 24$	$o([a, b]) = 9$
$McL{:}2$	$2B, 3B, 22$	$o((ab)^2(abab^2)^2ab^2) = 24$
$He{:}2$	$2B, 6C, 30$	none
$Suz{:}2$	$2C, 3B, 28$	none
$O'N{:}2$	$2B, 4A, 22$	none
$Fi_{22}{:}2$	$2A, 18E, 42$	none
$HN{:}2$	$2C, 5A, 42$	none
$Fi_{24}'{:}2$	$2C, 8D, 29$	none

5 New representations from old

There are many techniques available for obtaining new representations for a group from old ones. The basic method, which was the rationale behind the original development of the Meat-axe [22], is to tensor two matrix representations (over the same field) together, and then chop up the result into irreducibles. This enables many representations in the same characteristic as the original to be constructed. Various technical refinements can be used to extend the range of this basic technique. For example, use of symmetric and exterior squares, and other higher symmetrized powers, in addition to tensor products. The ideas of condensation, exploited by Ryba [28], Lux and Wiegelmann [14], and others, can also be used here.

A matrix representation will yield a permutation representation by calculating the action of the group on an orbit of vectors (or 1-dimensional subspaces, or k-dimensional subspaces, or any other convenient objects). A permutation representation can be reduced modulo any prime and chopped up with the Meat-axe into irreducibles—in this context the condensation method really comes into its own (see for example [10], among many others). These two ideas together enable one to change characteristic—that is, given a representation of G over a field of characteristic p, obtain one over a field of characteristic q.

Some of these techniques can change the group being represented. For example, the tensor product of two faithful irreducible representations of a double cover $2{}^{\cdot}G$ will represent only G, since the central involution acts as the scalar $(-1)\times(-1) = +1$. Another useful example here is the following: to obtain a representation of $12{\cdot}M_{22}$, take the tensor product of a representation of $3{\cdot}M_{22}$ and a representation of $4{\cdot}M_{22}$. Similarly, if an orbit of subspaces is permuted in a matrix representation, then in the resulting permutation representation the scalars act trivially.

There are two other important ways of changing the group. Firstly, if H is a subgroup of G, and we can find words in our generators of G which give generators of H, then any representation of G can be restricted to H. Secondly, if we have a representation of G then we can construct a representation of $G.\langle\tau\rangle$, where τ acts as an outer automorphism of G. Examples are described in [31], [9], [33], among others. Essentially, given a set $\{g_i\}$ of standard generators for G, words in the g_i are found giving images h_i of g_i under an outer automorphism τ. Then a 'standard basis' method (see [22]) is used to find explicitly a matrix (or permutation) conjugating the g_i to the h_i.

6 Future developments

There is clearly a great deal of room for further development of this 'Atlas of group representations'. Firstly there are other representations of the sporadic groups which are not easy to obtain from the given ones, but which may be interesting in their own right. Secondly, it would be very nice to have characteristic 0 matrix representations. Of course, permutation representations can be made into characteristic 0 matrix representations, but they are often far too big to handle in this way. A number of examples exist in the literature, in varying degrees of explicitness (see for example, Conway and Wales [3], [4], Norton [19], [21], Margolin [15]), and some work has been done by Stephen Rogers [25] on integrating these and others into the 'Atlas of group representations'.

Thirdly, there are other interesting groups which could be included. For example, as the referee pointed out, the exceptional covers of generic groups are closely related to the sporadic groups, and should be included. Since receiving the referee's report, I have made siginificant progress on constructing representations of these groups, but plenty remains to be done in this area.

Another way of extending the database would be to include the maximal subgroups of the sporadic groups. This could be done by including a library of procedures which, given standard generators for a group G, would produce a representative of each class of maximal subgroups of G. Some work on this has already been done by Peter Walsh [34].

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