The 2-modular characters of Conway's third group Co_3

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Abstract

We determine the 2-modular character table of the third sporadic simple group of Conway, up to two ambiguities. In each of these cases we give the smallest possibility for the character, which is also very likely to be the correct answer.

1 Introduction

The sporadic simple group Co_3 of order $495,766,656,000 = 2^{10}.3^7.5^3.7.11.23$ was discovered by J. H. Conway (see [2]). This group can be obtained as a subgroup of the double cover $2{\cdot}Co_1$ of Conway's first group Co_1, which may be defined as the automorphism group of the 24-dimensional Leech lattice (see [3]). In [5] Curtis has classified certain sublattices (which he called $\mathcal{S}$-lattices) of the Leech lattice and found corresponding subgroups of $2{\cdot}Co_1$ preserving these sublattices. Conway's third group Co_3 is the stabilizer of a type 3 vector in the Leech lattice (see [5]). The fourteen conjugacy classes of maximal subgroups of Co_3 were determined by Finkelstein (see [7]).

The p-modular character tables for Co_3 for the primes $p = 7$, 11, and 23 have been determined (see [10]). In this paper we find the 2-modular character table of Co_3. There are two characters which we were unable to prove correct, although we have very strong evidence for them.

Our methods are computational, making use mainly of the 'Meat-axe' written by R. A. Parker [8]. We also used GAP [13] for a few character calculations, as well as some extensions to the 'Meat-axe' written by M. Ringe of RWTH, Aachen. The calculations were performed partly on the IBM3090 in the Birmingham University Computer Centre, and partly on a SUN SPARCstation ELC provided by the School of Mathematics and Statistics in the University of Birmingham, with financial assistance from the Science and Engineering Research Council.

2 The blocks

Using the ordinary character table of Co_3 (see [4]), we calculate the 2-modular central characters. These central characters give the block distribution of characters. The ordinary characters are distributed in the following three blocks:

1. Block $B_1 = \{129536a, 129536b\}$ is of defect 1.

2. Block $B_2 = \{896a, 896b, 20608a, 20608b, 73600, 93312, 226688, 246400\}$ is of defect 3. The defect group is elementary abelian and of inertial index 21.

3. The principal block B_0 contains the remaining thirty two ordinary characters.

The theory of blocks of cyclic defect (see for example [1]) implies that the two ordinary representations $129536a$, $129536b$ which are in block B_1, have equal character values on the 2-regular classes, and that 129536 (the reduction modulo 2 of either one of them) is a 2-modular irreducible representation of Co_3.

3 Elementary use of the Meat-axe

To determine the decomposition matrices of the other two blocks, we use the Meat-axe, especially the method of condensation (see [12], and Section 4 below). We started with some 24×24 matrices over $GF(3)$ generating Co_3, as a subgroup of $2^{\cdot}Co_1$. Then using the Meat-axe program 'VP' ('vector permute'—see [14]) to find the action of Co_3 on a suitable orbit of vectors, we obtained permutations on 276 points also generating Co_3. Then we reduced the corresponding permutation representation modulo 2 and chopped this up with the Meat-axe to get $276 = 1 + 1 + 22 + 22 + 230$. In particular, we obtained two generators for Co_3 in $GL_{22}(2)$, as in Table 1.

Table 1: Generators for Co_3 in $GL_{22}(2)$

1101110001000001010000	0101000010111010111111
1111010111110100001011	0110010100011110110000
0000001000000100010101	0011010000111111010111
1111100110110001001110	0001101110001011010011
0101010000000010011101	1010010000100001011110
0000010000000100010101	1101000000001010100011
0010000000000100010101	1100101010001111010101
0001000011000000111111	1000110100110101010101
1110100100110100010011	0100110001010000000111
0000000000000110010101	1100000010100101010010
0000000000100100010101	0101110110011100000101
0110111111010011101111	0101111101010011111001
0000000000001100010101	1000010101010101010001
0000000000000100000101	0001010000111100100111
0000000001000100010101	0011010010111011001111
0000000000000100011101	0100110010110011111010
0001000110000010011010	1101011001111101100011
0000000000000000010101	0100101001001000100001
0000000000000101010101	1100101100001001110011
0000000000000100010100	0101110110010100000001
0000000000000100010111	0000001101111000101110
0000000000000100010001	1101101010101110000101
a	b

Table 2: Words for classes in Co_3

Class	Word
$9A$	$(ab^2)^2$
$9B$	ab
$11A$	$bab(abab^2)^2$
$15A$	$abab^2$
$15B$	$(ab)^2(abab^2)^2(ab^2)^2$
$21A$	$ab^2ab(abab^2)^2$
$23A$	$(ab)^2(abab^2)^2ab^2$

Then these two 2-modular irreducible representations (namely 22 and 230) can be used to produce some more irreducibles in the usual way with the Meat-axe. We find that

$$\begin{aligned}\Lambda^2(22) &= 1 + 230 \\ \Lambda^3(22) &= 22 + 22 + 1496 \\ 22 \otimes 230 &= 22 + 22 + 1496 + 3520.\end{aligned}$$

On restriction to the maximal subgroup $McL{:}2$, the ordinary characters $896a$, $896b$ remain irreducible. Using the 2-modular character table of $McL{:}2$ (see [9]) we can see that the latter remain irreducible on reduction modulo 2. It follows that $896a$ and $896b$ are two 2-modular irreducible representations of Co_3. Thus we have shown that 1, 22, 230, $896a$, $896b$, 1496, 3520 and 129536 are eight of the sixteen 2-modular irreducible representations of Co_3.

Before we look for the remaining irreducibles, we calculate the character values of the irreducibles obtained so far. To do this, we find representatives for all the 2-regular classes of Co_3 as words in a, b. The words for some classes are given in Table 2. The other 2-regular classes are powers of these. We then work out the character values on these classes using the program 'EV' of the Meat-axe which works out the eigenvalues of a matrix. The results are given in Table 3.

4 The method of condensation

The main method used in the following was condensation of permutation modules. Sometimes we also constructed the invariant subspaces which contain the required representations, by spinning up the corresponding subspaces in the condensed module under the group generators. This can be done using the uncondense program 'UK' of the Meat-axe. These methods have been explained in detail in [14] (see also [12] and [20]). We give here a brief summary.

Let G be a group, and V be a kG-module, where k is a finite field of characteristic p.

Table 3: The characters of some 2-modular irreducibles of Co_3

@	@	@	@	@	@	@	@	@	@	@	@	@	@	@	@
4	349	29	4	1											
95766656000	920	160	536	500	300	42	162	81	22	22	30	15	21	23	23
p power	A	A	A	A	A	A	A	A	A	A	AA	BB	AC	A	A
p' part	A	A	A	A	A	A	A	A	A	A	AA	BB	AC	A	A
1A	3A	3B	3C	5A	5B	7A	9A	9B	11A	B**	15A	15B	21A	23A	B**
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
22	-5	4	-2	-3	2	1	-2	1	0	0	0	-1	-2	-1	-1
230	14	5	2	5	0	-1	2	-1	-1	-1	-1	0	2	0	0
896	32	-4	-7	-4	1	0	2	-1	b11	**	2	1	0	-1	-1
896	32	-4	-7	-4	1	0	2	-1	**	b11	2	1	0	-1	-1
1496	-16	2	8	-4	-4	-2	-1	-4	0	0	-1	2	1	1	1
3520	-44	10	-8	-5	0	-1	1	1	0	0	1	0	-1	1	1
129536	-64	44	8	-14	-4	1	-1	-1	0	0	1	-1	1	0	0

Let H be a subgroup of G such that p does not divide $|H|$. Define the idempotent

$$e = \frac{1}{|H|} \sum_{h \in H} h \in kG.$$

Then $e.kG.e$ is a sub-algebra of kG known as a *Hecke algebra*.

From any kG-module V, we obtain an $e.kG.e$-module Ve. We say that Ve is *condensed* from V since Ve consists of the fixed points of the action of H on V. We call H the *condensation subgroup*. In this way we get a condensed module Ve such that $\dim(Ve) \approx \frac{\dim(V)}{|H|}$. Accordingly, it should be much easier to apply the 'Meat-axe' to Ve rather than to V. Moreover, any information about Ve which we obtain with the 'Meat-axe' gives rise to information about V. This can be seen using the following well-known result (see [12]).

Proposition 1 *Let $\chi_1, \chi_2, \ldots, \chi_r$ be the irreducible constituents of the module V, then the irreducible constituents of Ve are the non-zero members of the set $\{\chi_1 e, \chi_2 e, \ldots, \chi_r e\}$.*

The problem in practice is to prove that we have generators for the Hecke algebra. If $g_1, \ldots, g_r$ generate G, it does not necessarily follow that $eg_1e, \ldots, eg_re$ generate $e.kG.e$. We choose our group generators in such a way that we believe that $\langle eg_1e, \ldots, eg_re \rangle$ (which we call the *condensation algebra*) is the whole of the Hecke algebra $e.kG.e$, but it could conceivably be smaller. Thus we could have 'mirages'—subspaces of Ve invariant under the condensation algebra but not under the whole of the Hecke algebra. For this reason, condensation in general only provides lower bounds for the degrees of irreducible representations.

Sometimes we need more information about the constituents of V than straightforward condensation can provide. One method is to use the 'uncondense' program UK of the Meat-axe to calculate constituents of the permutation representation explicitly. Given an invariant subspace We of a condensed module Ve we can construct the corresponding invariant subspace W of the permutation module V as follows:

1. Use UK to express the basis vectors for We in terms of the basis vectors for V. Each basis vector for Ve is the sum of basis vectors in V over an orbit of H.

2. Spin up We under the group generators. That is, find the invariant subspace of V generated by We. In practice, we multiply the vectors we have so far by one of the generating permutations (using a version of 'MU' capable of multiplying non-square matrices by permutations), and use the Gaussian elimination programs 'EF' (echelon form), 'CL' (clean) and 'CE' (clean and extend) to put the whole collection of vectors into echelon form. Eventually, we obtain the required subspace that is invariant under the group generators. In particular, this gives us the degree of the representation, namely $\dim(W)$.

3. If we want to construct the matrices for G in this representation, we can apply the generators to our invariant subspace (using 'MU') and use 'CL' to write the images of the basis vectors as linear combinations of the basis vectors.

5 Condensation of some permutation modules

In this section we describe the application of the condensation method. The primitive permutation representations of Co_3 on 276, 11178, 37950, 48600, 128800, 170775 and 655776 points were found by using the Meat-axe program 'VP' as described above. Now they can be condensed using a suitable condensation subgroup. The permutation characters are found by inducing up the trivial character of the relevant maximal subgroups. These permutation characters are shown in Table 4, together with the permutation character of degree 1536975 which is used in the next section.

Using the ordinary character table of Co_3 (see [4]) we can see that these characters decompose as sums of the ordinary irreducible characters of Co_3 as follows.

$$
\begin{array}{lcrcl}
\mathrm{Pm1} & = & 276 & = & 1 + 275 \\
\mathrm{Pm2} & = & 11178 & = & 1 + 23 + 275 + 2024 + 8855 \\
\mathrm{Pm3} & = & 37950 & = & 1 + 275 + 275 + 5544 + 8855 + 23000 \\
\mathrm{Pm4} & = & 48600 & = & 1 + 23 + 253b + 275 + 2024 + 5544 + 8855 + 31625c \\
\mathrm{Pm5} & = & 128800 & = & 1 + 275 + 5544 + 8855 + 23000 + 91125 \\
\mathrm{Pm6} & = & 170775 & = & 1 + 275 + 7084 + 8855 + 23000 + 57960 + 73600 \\
\mathrm{Pm7} & = & 655776 & = & 1 + 275 + 2024 + 8855 + 23000 + 23000 + 57960 \\
 & & & & \quad + 73600 + 91125 + 129536b + 246400 \\
\mathrm{Pm8} & = & 1536975 & = & 1 + 23 + 253b + 2(275) + 2(2024) + 4025 + 2(5544) + 3(8855) + \\
 & & & & \quad + 2(23000) + 31625a + 3(31625c) + 31878 + 57960 + 73600 + \\
 & & & & \quad + 91125 + 2(125936b) + 177100 + 184437 + 2(221375)
\end{array}
$$

The permutation representation on 276 points is the only one that we can chop up directly using the Meat-axe, as the others are too big. Therefore we use the method of 'condensation' to find some more 2-modular irreducible representations.

We take our two generators a and b for Co_3, and find words to get a suitable condensation subgroup K such that the order of K is not divisible by 2. In this section we use a Sylow 5-subgroup K of order 125.

5.1 The permutation representation on 276 points

We first condense the first permutation module Pm1 over the subgroup K of order 125 to get a condensed module M1 of dimension 8. Although we do not need to condense Pm1 to find new irreducibles, yet we did so to identify the condensed irreducibles corresponding to the 2-modular irreducibles. M1 is chopped up using the Meat-axe as follows

$$\mathrm{M1} = 8 = 1 + 1 + 2a + 2a + 2b$$

and therefore we have the following correspondence.

Constituent of Pm1:	1	22	230
Constituent of M1:	1	$2a$	$2b$
Multiplicity:	2	2	1

Table 4: Permutation characters of Co_3

$1A$	$2A$	$2B$	$3A$	$3B$	$3C$	$4A$	$4B$	$5A$	$5B$	$6A$	$6B$	$6C$	$6D$	$6E$	$7A$
276	36	12	6	15	0	16	8	1	6	6	0	3	3	0	3
11178	378	66	0	81	0	6	30	3	13	0	0	9	3	0	6
37950	750	198	15	105	0	130	42	0	15	15	3	9	9	0	3
48600	1080	0	0	162	0	0	48	0	20	0	0	18	0	0	6
128800	1120	232	10	91	28	160	32	0	10	10	4	7	7	4	0
170775	631	495	351	135	0	31	15	25	0	31	19	7	9	0	3
655776	2016	792	486	135	72	96	32	1	6	6	36	3	9	0	2
1536975	7695	495	0	567	0	15	111	0	20	0	0	27	9	0	6

$8A$	$8B$	$8C$	$9A$	$9B$	$10A$	$10B$	$11A$	$11B$	$12A$	$12B$	$12C$	$14A$	$15A$	$15B$
2	6	2	0	3	1	2	1	1	4	2	1	1	1	0
12	0	4	0	0	3	1	2	2	0	0	3	0	0	1
4	16	4	0	3	0	3	0	0	7	3	1	1	0	0
0	0	4	0	0	0	0	2	2	0	0	0	2	0	2
0	8	4	1	1	0	2	1	1	4	2	1	0	0	1
7	7	3	9	0	1	0	0	0	7	3	1	1	1	0
8	0	4	0	0	1	2	0	0	0	2	3	0	1	0
3	3	3	0	0	0	0	0	0	0	0	3	2	0	2

$18A$	$20A$	$20B$	$21A$	$22A$	$22B$	$23A$	$23B$	$24A$	$24B$	$30A$
0	1	1	0	1	1	0	0	2	0	1
0	1	1	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	1	1	0
0	0	0	0	0	0	1	1	0	0	0
1	0	0	0	1	1	0	0	0	2	0
1	1	1	0	0	0	0	0	1	1	1
0	1	1	2	0	0	0	0	2	0	1
0	0	0	0	0	0	0	0	0	0	0

Table 5: Condensation of Pm2

Constituent of Pm2:	1	22	230	1496	7084
Constituent of M2:	1	$2a$	$2b$	$8a$	56
Multiplicity:	6	8	4	2	1

Of course, $1, 2a, 2b$ are the fixed spaces of K in the 2-modular irreducibles $1, 22$ and 230 respectively. We distinguish $2a$ from $2b$ by using a nullity fingerprint as described in [8].

5.2 The permutation representation on 11178 points

Now we take the permutation module Pm2 on 11178 points. We condense Pm2 over the same Sylow 5-subgroup K to get a condensed module M2 of dimension 102. M2 is chopped up as follows:

$$\mathrm{M2} = 102 = 6(1) + 8(2a) + 4(2b) + 2(8) + 56.$$

Now the permutation character 11178 decomposes as a sum of the ordinary characters as follows:

$$\mathrm{Pm2} = 11178 = 1 + 23 + 275 + 2024 + 8855.$$

Using the known character values (Table 3) we see that the following relations hold on the 2-regular classes:

$$\begin{aligned} 2024 &= 2(1) + 3(22) + 2(230) + 1496 \\ 8855 &= 1 + 2(22) + 230 + 1496 + 7084 \end{aligned}$$

where 7084 denotes the ordinary irreducible of degree 7084 for Co_3. By calculating the fixed spaces of K on the known representations, we find the correspondence between the constituents of the condensed module M2 and the irreducibles of the original module Pm2. This correspondence is given in Table 5.

We have proved already that 1496 is a 2-modular irreducible representation in Co_3. Moreover, 7084 lifts to characteristic 0, so is a 2-modular irreducible representation and the character values can be read off from the ordinary character table.

5.3 The permutation representation on 37950 points

Next we take the permutation representation on 37950 points and condense it over the same Sylow 5-subgroup K of order 125. The resulting condensed module M3 has dimension 318 and is chopped up to irreducibles as follows:

$$\mathrm{M3} = 318 = 8(1) + 11(2a) + 8(2b) + 2(8) + 2(28) + 56 + 72a + 72b$$

Table 6: Condensation of Pm3

Constituent of Pm3:	1	22	230	1496	3520	7084	9372a	9372b
Constituent of M3:	1	2a	2b	8a	28	56	72a	72b
Multiplicity:	8	11	8	2	2	1	1	1

Now using 'UK' as described in Section 4 we prove that the two new 2-modular irreducible characters of Co_3 corresponding to the condensed modules 72a and 72b each have dimension 9372. Table 6 gives the correspondence between the irreducibles of the condensed module M3 and the irreducibles of the permutation module Pm3.

The permutation character 37950 can be written as a sum of ordinary irreducibles as indicated below:

$$\mathrm{Pm3} = 37950 = 1 + 2(275) + 5544 + 8855 + 23000.$$

The ordinary irreducibles reduced modulo 2 break up as follows:

$$\begin{aligned} 1 &= 1 \\ 23 &= 1 + 22 \\ 275 &= 1 + 22 + 22 + 230 \\ 5544 &= 1 + 1 + 3(22) + 2(230) + 1496 + 3520 \\ 8855 &= 1 + 2(22) + 230 + 1496 + 7084 \end{aligned}$$

Hence 23000 contains all the remaining constituents, so from the condensation we obtain

$$23000 = 2(1) + 2(22) + 3(230) + 3520 + 9372a + 9372b.$$

Using the following relation between ordinary characters on the 2-regular classes of Co_3,

$$23000 + 23 = 253 + 3520 + 9625a + 9625b,$$

we obtain (without loss of generality)

$$9625a = 9372a + 23 + 230$$

and

$$9625b = 9372b + 23 + 230,$$

so we can work out the character values for the 2-modular irreducibles 9372a and 9372b.

5.4 The permutation representation on 48600 points

The character of the permutation module Pm4 on 48600 points can be written as

$$\mathrm{Pm4} = 48600 = 1 + 23 + 253b + 275 + 2024 + 5544 + 8855 + 31625c.$$

Also the following relation holds on the 2-regular classes:

$$31625c = 1771 + 3520 + 7084 + 9625a + 9625b.$$

Therefore, we can assume that condensation of this module will yield no new information.

Table 7: Condensation of Pm5

Constituent of Pm5:	1	22	230	1496	3520	7084	9372a	9372b	38456
Constituent of M5:	1	2a	2b	8a	28	56	72a	72b	304
Multiplicity:	14	18	13	4	3	2	3	3	1

5.5 The permutation representation on 128800 points

The permutation module Pm5 on 128800 points can be written as

$$\text{Pm5} = 128800 = 1 + 275 + 5544 + 8855 + 23000 + 91125.$$

We have the following relation on the 2-regular classes:

$$91125 = 1771 + 3520 + 7084 + 2(9625a) + 2(9625b) + 40250.$$

We condense this permutation module Pm5 over the same Sylow 5-subgroup K to get a 1040-dimensional condensed module M5 which is chopped up as follows:

$$1040 = 14(1) + 18(2a) + 13(2b) + 4(8) + 3(28) + 2(56) + 3(72a) + 3(72b) + 304.$$

The above decomposition gives an indication that there is a new 2-modular irreducible representation of degree at least 38456, corresponding to the condensed module of dimension 304. The correspondence between the different constituents is given in Table 7.

Unfortunately, we were unable to prove, as we did before, that 38456 is the exact dimension for the new 2-modular irreducible representation. Nevertheless, the evidence of the condensation is very strong, and we have a lower bound of 38456 and an upper bound of 40250 for the dimension.

5.6 The permutation representation on 170775 points

The permutation character 170775 decomposes as the sum of the following ordinary irreducibles:

$$\text{Pm6} = 170775 = 1 + 275 + 7084 + 8855 + 23000 + 57960 + 73600.$$

The ordinary irreducible character 73600 is in the block B_2 of defect 3, while the rest of the ordinary irreducibles are in the principal block B_0. The following relation holds on the 2-regular classes

$$57960 = 1 + 1771 + 3520 + 2(7084) + 2(9625a) + 2(9625b).$$

This relation shows that the only possible new 2-modular irreducibles are in 73600. In fact, it turns out that 73600 remains irreducible mod 2. Rather than prove this here, we will use the permutation representation of degree 655776, in the next section.

Table 8: Condensation of Pm7

Constituent of Pm7:	1	22	230	1496	3520	7084	9372a	9372b
Constituent of M7:	1	2a	2b	8a	28	56	72a	72b
Multiplicity:	22	26	21	5	4	4	6	6

38456	19712	73600	896a	896b	131584	129536
304	160	584	8b	8c	1056	1032
1	2	2	1	1	1	1

5.7 The permutation representation on 655776 points

The permutation character 655776 decomposes as

$$\begin{aligned} \text{Pm7} = 655776 &= 1 + 275 + 2024 + 8855 + 23000 + 23000 + 57960 + \\ &+ 73600 + 91125 + 129536b + 246400. \end{aligned}$$

Hence 73600 is one of the ordinary irreducibles of this permutation character as well. The ordinary irreducible 129536b was proved in Section 2 to be a 2-modular irreducible in Co_3. Restricting 129536 to the condensation subgroup K (the Sylow 5-subgroup) and calculating the fixed space in 129536 gives a dimension 1032. The condensation of all the ordinary irreducibles in the above permutation module Pm7 is already known except for 73600 and 246400. Also, the characters 73600 and 246400 are in the block B_2 of defect 3. We condense the permutation module Pm7 on 655776 over the same Sylow 5-subgroup K to get a condensed module M7 of dimension 5252. This condensed module M7 is chopped up as follows.

$$\begin{aligned} 5252 &= 22(1) + 26(2a) + 21(2b) + 5(8a) + 4(28) + 4(56) + 6(72a) + 6(72b) + \\ &+ 2(160) + 304 + 1032 + 2(584) + 8b + 8c + 1056. \end{aligned}$$

The correspondence between the irreducibles in the condensed module and the original module is given in Table 8.

This shows that 73600 remains irreducible mod 2. In the next section we show that the representations 19712 and 131584 both exist.

6 The Block B_2 of defect 3

The ordinary characters in block B_2, which is of defect 3, satisfy the following relations on the 2-regular classes:

$$\begin{aligned} 896a + 20608b &= 896b + 20608a & (1) \\ 896a + 93312 &= 20608a + 73600 & (2) \\ 896b + 93312 &= 20608b + 73600 & (3) \\ 896a + 246400 &= 20608a + 226688 & (4) \\ 896b + 246400 &= 20608b + 226688 & (5) \\ 93312 + 226688 &= 246400 + 73600 & (6) \end{aligned}$$

Table 9: The decomposition matrix D_2 of block B_2

	$896a$	$896b$	19712	73600	$X \geq 131584$
$896a$	1	0	0	0	0
$896b$	0	1	0	0	0
$20608a$	1	0	1	0	0
$20608b$	0	1	1	0	0
73600	0	0	0	1	0
93312	0	0	1	1	0
226688	$a \leq 1$	$a \leq 1$	$b \leq 1$	$c \leq 1$	1
246400	$a \leq 1$	$a \leq 1$	$1 + b \leq 2$	$c \leq 1$	1

Using relation (1) we deduce that as 2-modular characters $896a < 20608a$ and $896b < 20608b$, and therefore $20608a - 896a = 19712$ is a 2-modular character. Using condensation we have proved in Section 5.7 that 19712 is a lower bound for the degree of this representation. Hence 19712 is a 2-modular irreducible representation in Co_3.

Using the above relations between the ordinary characters in block B_2 and the information we have got so far we can write down the decomposition matrix of this block with some ambiguities on some entries. This is shown in Table 9.

The following relation holds for ordinary characters:

$$253 \otimes 896a = 226688.$$

This implies that

$$(1 + 22 + 230) \otimes 896a = 226688$$

on the 2-regular classes. Moreover,

$$22 \otimes 896a = 19712.$$

Therefore,

$$896a + 19712 + 230 \otimes 896a = 226688.$$

Hence,

$$896a < 226688, \quad 19712 < 226688 \quad \text{and} \quad \overline{896a} = 896b < 226688.$$

The above results imply that $226688 - 896a - 896b - 19712 = 205184$ is a 2-modular character. Using relation (6) above we can deduce that $896a, 896b$ and $2(19712)$ are contained in 246400. Thus we have $a = 1$ and $b = 1$ in Table 9.

To show that $c = 1$, we return to considering the permutation representation Pm7. The part of Pm7 in the block B_2 is just $73600 + 246400$, and the corresponding condensed module M7 has constituents $8b + 8c + 2(160) + 2(584) + 1056$ in this block. Now we can show with the Meat-axe that M7 has a unique submodule of dimension 584 (corresponding to the ordinary irreducible 73600). Similarly there is a unique submodule of codimension

584, which contains the submodule of dimension 584. Thus the image of 73600 in the 2-modular permutation representation is contained in the image of 246400. In other words, 73600 is a constituent of the reduction modulo 2 of 246400.

Remark 1 *An alternative method of showing that* $c \geq 1$ *is as follows. The part of* $22 \otimes 73600$ *in* B_2 *is* 19712*. Thus*

$$\mathrm{Hom}(22 \otimes 19712, 73600) = \mathrm{Hom}(19712, 22 \otimes 73600) \neq 0.$$

But the constituents of $22 \otimes 19712$ *in this block are*

$$2(896a) + 2(896b) + 19712 + 2(205184),$$

and so $73600 < 205184$.

7 The last irreducible

There remains one irreducible to be found, which is contained in the permutation module of degree 1536975, on the cosets of the subgroup $2^{4\cdot}A_8$. Using GAP [13] we see that the character of this representation can be expressed in terms of the ordinary irreducibles as

$$\begin{aligned}\mathrm{Pm8} \ = \ & 1 + 23 + 253b + 2(275) + 2(2024) + 4025 + 2(5544) + 3(8855) + \\ & + 2(23000) + 31625a + 3(31625c) + 31878 + 57960 + 73600 + \\ & + 91125 + 2(1259936b) + 177100 + 184437 + 2(221375)\end{aligned}$$

Using the relations which hold on the 2-regular classes, we obtain the following expression in terms of the 2-modular irreducibles.

$$\begin{aligned}\mathrm{Pm8} \ = \ & 57(1) + 77(22) + 56(230) + 17(1496) + 14(3520) + 12(7084) + 11(9372a) + \\ & + 11(9372b) + 3(38456) + 73600 + 2(129536) + 4(177100),\end{aligned}$$

where the last character 177100 denotes the reduction modulo 2 of the ordinary representation of that degree.

We cannot condense this representation with the Sylow 5-subgroup K, since it is too big. Thus we need to condense with a larger subgroup. The problem then is that some of the irreducibles condense to dimension 0. We overcome this by condensing with two different subgroups. We took the groups $K_1 \cong 3^4{:}5$ and $K_2 \cong 23{:}11$. The class distribution of the elements of K_1 is $(1A, 3A^{20}, 3B^{60}, 5B^{324})$, so the condensed dimensions of the 2-modular irreducibles we have found so far are as in Table 10.

Now Pm8 condensed over K_1 breaks up as

$$73(1){+}101(2a){+}72(2b){+}18(8a){+}16(20a){+}19(20b){+}19(20c){+}7(80){+}188{+}4(216){+}2(320).$$

Similarly Pm8 condensed over K_2 breaks up as

$$73(1) + 21(6) + 18(14) + 16(28) + 19(37a) + 19(37b) + 7(152) + 290 + 4(348) + 2(512).$$

Table 10: Condensation over K_1 and K_2

Degree	Dimension over K_1	Dimension over K_2
1	1	1
22	2	0
230	2	0
896	4	3
1496	0	6
3520	8	14
7084	20	28
9372	20	37
19712	40	78
38456	80	152
73600	188	290
129536	320	512
131584	336	522

Therefore it is very likely that the last irreducible condenses to dimension 216 over K_1 and 348 over K_2. On this assumption we obtain the last 2-modular character as

$$177100 - 4(1) - 6(22) - 4(230) - 1496 - 3520 - 7084 - 2(9372a) - 2(9372b) - 38456$$

on the 2-regular classes. This has degree 88000.

8 The indicators

Every 2-modular self-dual irreducible module supports an invariant symplectic form. Some will also support an invariant quadratic form. Following [9], we use the symbol $+$ to denote that the 2-modular irreducible representation supports a non-zero invariant quadratic form; if not, we use the symbol $-$. It is often very difficult to determine, theoretically, whether a 2-modular representation supports an invariant quadratic form or not, so we use computer calculations to solve this problem.

The representations 1, 7084, 73600 and 129536 lift to ordinary representations of the same degree, which have Schur indicator $+$, so support invariant quadratic forms. Other cases required computer calculations using the method which was explained in detail in [14]. Here is a brief explanation of that method.

Using the programs of the Meat-axe, Standard-Base 'SB', Transpose 'TR' and Invert 'IV' (to get the dual representation) and Standard-Base again, we find a matrix P such that

$$P^{-1} g_i P = ({g_i}^T)^{-1}$$

for each group generator g_i. Hence $g_i P {g_i}^T = P$ and P is the matrix of a symplectic form invariant under Co_3.

Now a quadratic form q can be specified by giving the associated symplectic form, together with the values of q on a basis. Since all the basis vectors produced by 'SB' are in the same orbit under the group, there is just one possible quadratic form for each element of the field.

Each quadratic form may be represented by a matrix Q obtained by taking the bottom-left of P (i.e the part below or on the main diagonal), and adding a scalar matrix. We have to check whether the diagonal of $g_i Q {g_i}^T$ is equal to the diagonal of Q. If it is, for all the generators g_i of G, then the quadratic form represented by Q is invariant under G. Using the Meat-axe we proved that the representations 230, 1496 and 3520 have indicator $+$, while 22 has indicator $-$.

The remaining representations are too big to approach in this way, but those which are not in the principal block yield to a theoretical approach. In fact we prove a slightly more general result, which implies in particular that every self-dual irreducible outside the principal block has indicator $+$. First we set up the usual machinery of reduction modulo p. We have a group G, a field k of characteristic 0, a ring of integers R in k, a prime $\wp$ in R dividing p, a kG-module V, and an RG-lattice Λ such that $\Lambda/\wp\Lambda$ is a reduction modulo p of V. We use a theorem of Thompson which states that Λ can be chosen in such a way that any desired constituent is the unique top composition factor of $V' = \Lambda/\wp\Lambda$.

Theorem 1 *With the above notation, suppose $p = 2$ and V is irreducible, and V supports a non-degenerate G-invariant quadratic form q. Suppose that V' has no trivial constituents, U is a self-dual 2-modular irreducible for G, and V' has exactly one composition factor isomorphic to U. Then U supports a non-degenerate G-invariant quadratic form.*

Proof. We choose Λ so that $\Lambda/\wp\Lambda$ has U as unique top composition factor. By clearing denominators we can ensure that $q|_\Lambda : \Lambda \to R$, and by dividing by a suitable power of $\wp$ we can ensure that $q(\Lambda)$ is not contained in $\wp R$. Thus q induces a non-zero quadratic form $q' : V' \to F$, where $F = R/\wp R$ is a field of characteristic 2. Since V has no trivial constituents, the radical of q' coincides with the radical of the associated bilinear form, and q' induces a non-degenerate quadratic form q'' on $V'' = V'/rad(q')$. Moreover, U is the unique top composition factor of V''. Thus $U^* \cong U$ is the unique bottom composition factor of V''. But by assumption U occurs only once in V', so $V'' \cong U$ and the result follows.

Corollary 2 *Every 2-modular irreducible with indicator $-$ is in the principal block.*

Theorem 3 *Table 11 represents the 2-modular character table of Co_3, up to two ambiguities.*

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Table 11: The 2-modular character table of Co_3

;	@	@	@	@	@	@	@	@	@	@	@	@	@	@	@	@
	4	349	29	4	1											
	95766656000	920	160	536	500	300	42	162	81	22	22	30	15	21	23	23
p power		A	A	A	A	A	A	A	A	A	A	AA	BB	AC	A	A
p' part		A	A	A	A	A	A	A	A	A	A	AA	BB	AC	A	A
ind	1A	3A	3B	3C	5A	5B	7A	9A	9B	11A	B**	15A	15B	21A	23A	B**
+	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
-	22	-5	4	-2	-3	2	1	-2	1	0	0	0	-1	-2	-1	-1
+	230	14	5	2	5	0	-1	2	-1	-1	-1	-1	0	2	0	0
o	896	32	-4	-7	-4	1	0	2	-1	b11	**	2	1	0	-1	-1
o	896	32	-4	-7	-4	1	0	2	-1	**	b11	2	1	0	-1	-1
+	1496	-16	2	8	-4	-4	-2	-1	-4	0	0	-1	2	1	1	1
+	3520	-44	10	-8	-5	0	-1	1	1	0	0	1	0	-1	1	1
+	7084	10	19	-14	9	-1	0	4	-2	0	0	0	-1	0	0	0
o	9372	30	-15	6	-3	-3	-1	-3	0	0	0	0	0	-1	b23	**
o	9372	30	-15	6	-3	-3	-1	-3	0	0	0	0	0	-1	**	b23
+	19712	-160	-16	14	12	2	0	-4	-1	0	0	0	-1	0	1	1
?	38456?	-100	-46	8	6	-4	-2	8	-1	0	0	0	-1	1	0	0
+	73600	160	16	13	0	-5	2	4	1	-1	-1	0	1	-1	0	0
?	88000?	-20	-2	-32	0	0	3	-5	1	0	0	0	3	3	2	2
+	129536	-64	44	8	-14	-4	1	-1	-1	0	0	1	-1	1	0	0
+	131584	256	-32	-20	-16	4	-2	-2	1	2	2	-4	-2	1	1	1

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