An eightfold path to E_8

Robert A. Wilson

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Introduction

Finite-dimensional real reflection groups were classified by Coxeter [2]. In two dimensions, they are the dihedral groups, and in higher dimensions there are three infinite series, A_n, B_n and D_n in n dimensions (for each $n > 2$, although in fact $A_3 = D_3$), and six exceptional cases, H_3, H_4, F_4, E_6, E_7, E_8, where in each case the subscript indicates the dimension.

It is significant (I think) that the various finite series of reflection groups stop in dimensions 2, 4, and 8, precisely the dimensions of the complex numbers, quaternions, and octonions (Cayley numbers). Thus the 'end of the road' is reached for dihedral groups of order at least 12 in two dimensions, F_4 and H_4 in four dimensions, and E_8 in eight dimensions.

This significance perhaps lies in the fact that reflections are not just geometric concepts, but (at least in 2, 4 or 8 dimensions) inescapably algebraic concepts. It is a well-known fact, which we teach to undergraduates, that reflections in 2 dimensions can be expressed in terms of complex numbers. Similarly, reflections in 4 dimensions can be expressed in terms of quaternions, and in 8 dimensions in terms of octonions. In each case the reflection which negates the unit vector r and fixes everything perpendicular to r is given by the rule

$$x \mapsto -r\overline{x}r.$$

This map is usually called *reflection in* r.

The aim of this expository note is to bring many of these ideas together and to see how E_8 may be reached by many different paths. In Section n we consider the case of dimension 2^n. Thus we begin in Section 1 with a reminder of the description of dihedral groups in terms of complex numbers. Then in Section 2 we describe the 4-dimensional cases, first in terms of a real Euclidean space, then in terms of quaternions. Then in Section 3 we move on to E_8, in various manifestations as (a) 8-dimensional over $\mathbb{R}$, (b) 4-dimensional over $\mathbb{C}$, (c) 2-dimensional over $\mathbb{H}$, and (d) 1-dimensional over octonions.

1 Two dimensions

The 2-dimensional reflection groups are just the dihedral groups of order $2k$, consisting of k rotations about the origin, and k reflections in axes through the origin. Of particular interest are the cases $k = 2$ and $k = 3$, which are treated in Subsections 1.k below.

1.1 Complex numbers

Since the map $z \mapsto \overline{z}$ maps i to $-i$ while fixing 1, it is the map 'reflect in the direction of $\pm i$'. Similarly, 'reflect in the direction of ± 1' is expressed as $z \mapsto -\overline{z}$.

Moreover, if r is any complex number with $|r| = 1$ (that is $r\overline{r} = 1$), then the map $z \mapsto rz$ is rotation about the origin, such that 1 maps to r. Therefore we can obtain the map 'reflect in the direction of $\pm r$' as the composite of

1. rotate r to 1, by $z \mapsto \overline{r}z$;
2. reflect in 1, by $\overline{r}z \mapsto \overline{\overline{r}z} = \overline{z}r$;
3. rotate 1 back to r, by $\overline{z}r \mapsto r\overline{z}r$.

Thus it is the map

$$z \quad \mapsto \quad -r\overline{z}r. \tag{1}$$

Since the complex numbers are commutative, this can be simplified to $z \mapsto -r^2\overline{z}$, but I prefer not to do this, because then the above calculation goes through unchanged in the quaternions and even in the octonions.

More generally, for any non-zero complex number r, reflection in the direction of $\pm r$ can be expressed as the composite map

$$\begin{aligned} z &\mapsto r^{-1}z \\ &\mapsto \overline{r^{-1}z} = \overline{z}\,\overline{r}^{-1} = \overline{z}r/(\overline{r}r) \\ &\mapsto r\overline{z}r/(\overline{r}r). \end{aligned} \tag{2}$$

Now the product of any two reflections is a rotation, since

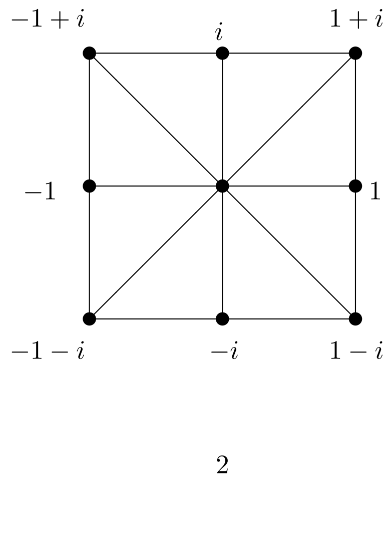
$$-s(-\overline{r\overline{z}r})s = s\overline{r}z\overline{r}s = (\overline{r}s)^2 z.$$

Moreover, the group structure on these rotations is given by multiplication in the complex numbers, since the composite of $z \mapsto az$ with $z \mapsto bz$ is $z \mapsto (ab)z$.

1.2 The dihedral group of order 8

In the general classification of reflection groups, the dihedral group of order 8 is denoted B_2. This is because it is the symmetry group of a square, which is the beginning of a series which includes the cube, hypercube and so on, whose reflection groups are B_3, B_4 and so on.

The four rotations of this dihedral group are given by multiplying by ± 1 or $\pm i$. Since reflection in $-r$ is the same as reflection in r, reflecting in ± 1 or $\pm i$ gives us only two reflections, and the other two are given by reflection in the two directions given by the unit vectors $(\pm 1 \pm i)/\sqrt{2}$. It is often convenient to avoid the factor of $\sqrt{2}$ by using $\pm 1 \pm i$ instead (see picture). Indeed, one could equally well use $(\pm 1 \pm i)/2$, in which case the picture would look slightly different.



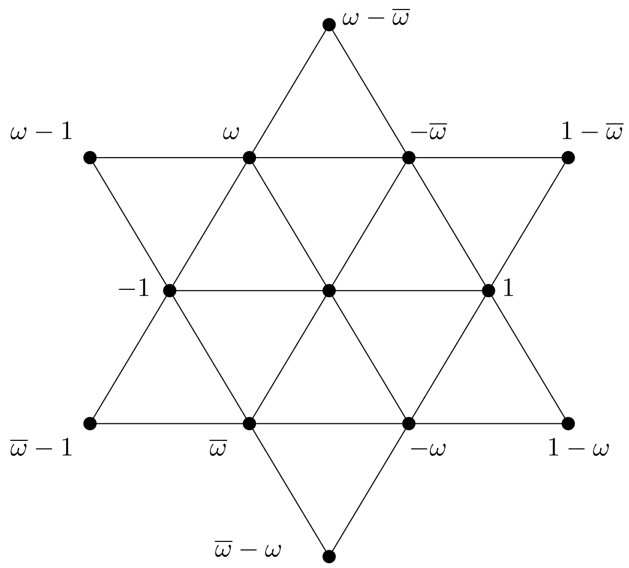
The reflecting vectors $\pm 1, \pm i, \pm 1 \pm i$ are known as *roots*. For obvious reasons, the vectors ± 1 and $\pm i$ are called *short roots* and $\pm 1 \pm i$ are *long roots*. Notice that these roots lie in the ring of *Gaussian integers*, $\mathbb{Z}[i]$. Moreover, the long roots can be obtained from the short roots by multiplying by $1 + i$.

Converting from $\mathbb{C}$ to $\mathbb{R}^2$ in the usual way, we can write the short roots as the vectors $(\pm 1, 0)$, $(0, \pm 1)$ and the long roots as $(\pm 1, \pm 1)$. In the alternative picture, we can write the short roots as $\pm 1 \pm i = (\pm 1, \pm 1)$, and the long roots as $\pm 2 = (\pm 2, 0)$ and $\pm 2i = (0, \pm 2)$.

1.3 The dihedral group of order 12

The dihedral group of order 12 also occupies a special place in the world of reflection groups. It is the symmetry group of a regular hexagon, and like squares and equilateral triangles, but unlike any other regular polygons, regular hexagons can tile the plane. This group is therefore give a special name, G_2.

Let $\omega = (-1 + \sqrt{-3})/2$ be a primitive cube root of 1 in the complex numbers. Then the rotations of G_2 are given by the powers of $\pm\omega$. Three of the six reflections are given by the same vectors, and the other three are given by $\pm\omega^a\sqrt{-1}$. As in the case of B_2, it is convenient to scale these to $\pm\omega^a(\omega - \overline{\omega}) = \pm\omega\sqrt{-3}$, or to $\pm\omega^a(\omega - \overline{\omega})/3 = \pm\omega\sqrt{-1/3}$. (See the picture.)



For completeness, let us note that there is a subgroup consisting of the three rotations by $1, \omega, \overline{\omega}$ and the three reflections in $\pm 1, \pm\omega, \pm\overline{\omega}$. This is a dihedral group of order 6, and is known as A_2. In both cases A_2 and G_2, the corresponding ring of complex integers is $\mathbb{Z}[\omega]$, often called the ring of *Eisenstein integers*.

2 Four dimensions

Even in three dimensions the situation is radically different from two dimensions. Instead of there being an infinite number of indecomposable reflection groups, there are only three, namely A_3 (the group of the regular tetrahedron), B_3 (the group of the cube or regular octahedron), and H_3 (the group of the regular dodecahedron or icosahedron). All three can be described in terms of quaternions, as subgroups of 4-dimensional groups A_4, B_4 and H_4 respectively.

Altogether there are five indecomposable reflection groups in four dimensions, namely A_4, B_4, D_4, F_4, and H_4. We shall see that F_4 contains both B_4 and D_4, while H_4 contains A_4 and D_4.

2.1 Quaternions

The non-commutative ring of (real) quaternions is $\mathbb{H} = \mathbb{R}[i,j,k]$ where

$$\begin{array}{rcl} i^2 = j^2 = k^2 & = & -1, \\ ij = -ji & = & k, \\ jk = -kj & = & i, \\ ki = -ik & = & j. \end{array} \tag{3}$$

If $q = a + bi + cj + dk$ then we write

$$\overline{q} = a - bi - cj - dk,$$

and then it is straightforward to compute that $\overline{q}q$ is equal to the square of the Euclidean length of q. In particular, if $q \neq 0$ then $q^{-1} = \overline{q}/(\overline{q}q)$. Moreover, since $(qr)^{-1} = r^{-1}q^{-1}$, we also have $\overline{qr} = \overline{r}.\overline{q}$.

Reflections in 4-dimensional Euclidean space can be expressed in terms of quaternion notation in just the same way as in the complex numbers, though taking extra care because of the non-commutativity. That is, reflection in 1 is the map

$$q \mapsto -\overline{q}$$

So if $\overline{r}r = 1$ then reflection in r is the map

$$q \mapsto -r\overline{\overline{r}q} = -r\overline{q}r.$$

In the 4-dimensional cases F_4 and H_4 suitable sets of reflecting vectors (not necessarily of norm 1) can be found inside respectively the Hurwitz integers and the icosians, which are well-known rings of quaternions that we describe below.

These 4-dimensional reflection groups may again be described by *root systems*, in which each *root* is one of the reflecting vectors, scaled appropriately.

2.2 Complex reflection groups

We shall also describe these root systems in terms of $\mathbb{C}^2$, a kind of 'halfway house' between the real and quaternionic descriptions. The real reflections then need a new interpretation in this context. Recall that reflection in the real vector r is the map

$$v \mapsto v - 2\frac{(v,r)}{(r,r)}r,$$

where (v,r) denotes the inner product $\sum v_i r_i$ of the vectors $v = (v_i)$ and $r = (r_i)$. (This map is linear, and fixes v if $(v,r) = 0$, and maps r to $-r$, so must be the required reflection.) Similarly if v and r are complex vectors, with inner product $(v,r) = \sum_i v_i \overline{r_i}$, then the map

$$v \mapsto v - 2\frac{(v,r)}{(r,r)}r$$

again fixes v if $(v,r) = 0$, and maps r to r. But this map is linear over $\mathbb{C}$, so negates a complex 1-space, that is a real 2-space, so is not a reflection in the real sense.

To recover the real reflection, we have to make the inner products real. We define a *real* inner product to be any convenient real multiple of the real part of the *complex* inner product. Thus

$$\begin{aligned} v &\mapsto v - 2\frac{\Re(v,r)}{(r,r)}r \\ &= v - \frac{(v,r)+\overline{(v,r)}}{(r,r)}r \end{aligned} \tag{4}$$

This map still takes r to $-r$, but now takes ir to itself.

2.3 Integer quaternions, B_4 and C_4

The 'obvious' analogue in the quaternions of the Gaussian integers is the ring

$$\mathbb{Z}[i,j,k] = \{a + bi + cj + dk \mid a,b,c,d \in \mathbb{Z}\}.$$

The units of this ring are just the elements $\pm 1, \pm i, \pm j, \pm k$, which form the well-known quaternion group Q_8. This ring is invariant not only under reflections in these units ('short roots'), but also under reflections in vectors of norm (i.e. squared length) 2, such as $1 + i$. In total there are 24 such 'long roots' (6 choices for which two of $1, i, j, k$ to take, and 4 choices of signs). Together these 8 short roots and 24 long roots form the 'root system' of type B_4.

Converting to $\mathbb{R}^4$ in the obvious way, these roots become vectors of shape $(\pm 1, 0, 0, 0)$ and $(\pm 1, \pm 1, 0, 0)$, up to permutations of the coordinates. The corresponding reflection group is generated by the corresponding 4 reflections in short roots and 12 reflections in long roots. Reflection in a short root just changes the sign of one of the four coordinates, while reflection in a long root swaps two coordinates (with or without changing their signs), so the full group has shape $2^4.S_4$.

Just as in the case B_2, we can interchange the roles of long and short roots, by multiplying the short roots by 2. This time, however, the numbers of long and short roots are different, so we obtain a different sort of root system, even though the reflection group is still the same. This new root system is called C_4, and has 12 short roots of the shape $(\pm 1, \pm 1, 0, 0)$ and 8 long roots of the shape $(\pm 2, 0, 0, 0)$.

Cutting down to three dimensions (the 3-space of pure imaginary quaternions is the most convenient one to take), we obtain a root system of type B_3 in which the roots are 6 of shape $(\pm 1, 0, 0)$ and 12 of shape $(\pm 1, \pm 1, 0)$. With respect to a natural coordinate system for the cube, the short roots are the midpoints of the six faces, and the long roots are the midpoints of the edges. Similarly the C_3 root system can be described by saying that the long roots are the vertices of a regular octahedron, and the short roots are the midpoints of the edges.

2.4 Hurwitz quaternions, D_4 and F_4

If we multiply the short roots of B_4 by $1 + i$ (on either the left or the right) we get 8 of the 24 long roots, namely $\pm 1 \pm i$ and $\pm j \pm k$. To get short roots corresponding to the other long roots of B_4, we would have to multiply by $(1 - i)/2$, and then we would obtain 16 roots such as

$$(1 + j)(1 - i)/2 = (1 - i + j + k)/2.$$

Indeed it does not matter whether we multiply on the left or the right, in either case we end up with all 16 roots $(\pm 1 \pm i \pm j \pm k)/2$. It is a remarkable fact that the resulting set of 24 short roots is closed under quaternion multiplication, and hence spans a ring known as the Hurwitz quaternions, which is still 'integral' in the sense that the norms of the elements are integers.

This ring may be obtained by adjoining a single element such as

$$\omega = \frac{1}{2}(-1 + i + j + k).$$

Since ω is of the form $\frac{1}{2}(-1+\sqrt{-3})$, we have $\omega^2 = \overline{\omega} = -1 - \omega$ and $\omega^3 = 1$, and is easy to see that the product of ω with any of i, j, k is another element with $\pm\frac{1}{2}$ in each of the four coordinates. Hence the ring consists of all elements $a + bi + cj + dk$ where either $a, b, c, d \in \mathbb{Z}$ or

$$a - \frac{1}{2}, b - \frac{1}{2}, c - \frac{1}{2}, d - \frac{1}{2} \in \mathbb{Z}.$$

Thus in particular the unit elements (that is the elements with norm 1) are now $\pm 1, \pm i, \pm j, \pm k$ together with $\frac{1}{2}(\pm 1 \pm i \pm j \pm k)$, making 24 in all. These units form a group variously known as $2.A_4$ or $SL_2(3)$ or the binary tetrahedral group.

It follows that the set of 48 roots we have constructed forms a root system, in the sense that it is closed under reflection in any one of the roots. It is called the root system of type F_4. The short roots are the 24 units just listed, and the 24 long roots are their multiples by $1 + i$, and are identical to the long roots of B_4. The 24 short roots on their own form what is known as the root system of type D_4. Similarly, the 24 long roots also form a root system of type D_4.

Converting to $\mathbb{R}^4$ as before, the roots of D_4 may be taken as all 24 vectors of shape $(\pm 1, \pm 1, 0, 0)$. Or, on a different scale, the 8 vectors of shape $(\pm 2, 0, 0, 0)$ together with the 16 of shape $(\pm 1, \pm 1, \pm 1, \pm 1)$. The roots of F_4 are of two different lengths, and consist of the two copies of D_4 just mentioned. They can be scaled so that either of them gives the short roots and the other one gives the long roots. In terms of quaternions, multiplication by $1+i$ converts from one version to the other. To obtain B_4, take all of the long roots $(\pm 1, \pm 1, 0, 0)$ of F_4 and the eight short roots of shape $(\pm 1, 0, 0, 0)$. Similarly, C_4 may be obtained from all the short roots of F_4 and a suitable set of 8 long roots. For example in the other description of F_4, we may take the short roots to be of shape $(\pm 1, \pm 1, 0, 0)$, and the 8 long roots of C_4 to be of shape $(\pm 2, 0, 0, 0)$. Multiplication by $1 + i$ gives different descriptions of the root systems B_4 and C_4.

The full reflection group of F_4 has shape $2.(A_4 \times A_4).2.2$ in which the central 2 is negation of the whole 4-space, modulo which the two copies of A_4 are left- and right-multiplication by the units. Then the maps $q \mapsto \frac{1}{2}(1+i)q(1+i)$ and $q \mapsto \overline{q}$ extend this to the whole group.

There are also descriptions of all these roots systems in terms of 2-dimensional complex spaces. The system of type F_4 can be described by taking the short roots to be 8 of shape $(\pm 1 \pm i, 0)$ and 16 of shape (i^a, i^b), and the long roots to be their multiples by $1 + i$. Similar descriptions of the other systems can be obtained from their embeddings in F_4.

Indeed, D_4 and F_4 also have nice descriptions over the Eisenstein integers. We take 6 roots $\pm\omega^a(\omega - \overline{\omega}, 0)$ and 18 roots $\pm(\omega^a, \omega^b\sqrt{2})$ to be the roots of D_4, that is the short roots of F_4. Then the long roots of F_4 are the sums of pairs of short roots which are perpendicular in the real sense. These can be calculated to be the vectors obtained from the short roots by multiplying by $\sqrt{2}$ and swapping the two coordinates, thus $\pm(2\omega^a, \omega^b\sqrt{2})$, and $\sqrt{2}(0, \omega^a - \omega^b)$ with $a \neq b$. Then B_4 may be obtained from F_4 by taking all the long roots but only the 8 short roots $\pm(\omega - \overline{\omega}, 0)$ and $\pm(1, \omega^a\sqrt{2})$. Similarly C_4 is obtained by taking all the short roots but only the 8 long roots $\pm\sqrt{2}(0, \omega - \overline{\omega})$ and $\pm(2\omega^a, \sqrt{2})$.

2.5 Icosians, A_4 and H_4

There is a similar description of the reflection group of type H_4, obtained by extending D_4 in a different way. We can imagine the D_4 root system by looking at the quaternions $(\pm i \pm j \pm k)/2$ which are added to $\pm 1/2$ as the vertices of a unit cube. Since a cube can be inscribed in a regular dodecahedron, one might wonder what would happen if one used the extra vertices of the dodecahedron in the same way as those of the cube. Indeed, there are two such dodecahedra, and in one case the extra 12 vertices are $(\pm\sigma i \pm \tau j)/2$ and its images under $i \mapsto j \mapsto k \mapsto i$, where $\sigma = \frac{1}{2}(\sqrt{5}-1)$ and $\tau = \frac{1}{2}(1+\sqrt{5})$. In the other case, which we shall use, σ and τ are interchanged.

Closing under reflections in the roots we quickly see that there are 120 roots, which are the 24 roots of D_4 together with the images of $(\pm 1 \pm \tau i \pm \sigma j)/2$ under even permutations of $\{1, i, j, k\}$. Again we find the remarkable fact that these 120 roots are closed under quaternion multiplication. They form the group of units of the so-called *icosian ring*

$$\mathbb{Z}[i, j, k, \frac{1}{2}(1+i+j+k), \frac{1}{2}(1+\tau i+\sigma j)].$$

This group of order 120 is a double cover of A_5, also known as $SL_2(5)$ or the binary icosahedral group.

The full reflection group of type H_4 has shape $2.(A_5 \times A_5).2$ in which we see left- and right-multiplications by this group of units, together with the map $q \mapsto \overline{q}$ again.

Translating back to $\mathbb{R}^4$, we see that the roots of H_4 are the even permutations of $(\pm 2, 0, 0, 0)$, $(\pm 1, \pm 1, \pm 1, \pm 1)$, and $(0, \pm 1, \pm\sigma, \pm\tau)$. Inside H_4 there is a copy of A_4 consisting of 20 roots

$$(\pm 2, 0, 0, 0), (0, \pm 2, 0, 0), \pm(\pm 1, \pm 1, 1, 1), \pm(\pm 1, 0, \tau, \sigma), \pm(0, \pm 1, \sigma, \tau).$$

3 Eight dimensions

There are innumerable ways to make E_8. The rest of this note is devoted to describing a few of these constructions in some detail.

In the 8-dimensional case E_8, we can similarly obtain a complete set of reflecting vectors inside the Dickson–Coxeter integral octonions inside $\mathbb{O}$.

3.1 Real E_8

Probably the most common description of E_8 is to take first the D_8 roots which are all permutations of $(\pm 1, \pm 1, 0, 0, 0, 0, 0, 0)$, so there are 112 of them, and then adjoin 128 roots of the form $\frac{1}{2}(\pm 1, \pm 1, \pm 1, \pm 1, \pm 1, \pm 1, \pm 1, \pm 1)$. According to taste, one can either take the sign combinations in which there are an even number of $-$ signs, or those with an odd number.

3.2 A Gaussian version of E_8

One of the simplest constructions of E_8 starts from the B_2 root system, consisting of four long roots which are the corners of a square, and four short roots which are the mid-points of the edges. We take 16 roots $(2r, 0, 0, 0)$ where r is a short root of B_2, and $6.4.4 = 96$ roots $(r_1, r_2, 0, 0)$ where r_i are long roots of B_2, and 128 roots (r_1, r_2, r_3, r_4) where the r_i are short roots of B_2 and their sum lies in the lattice spanned by the long roots. (Alternatively, this construction can be twisted by taking the sum *not* to lie in this lattice).

The B_2 lattice may be embedded in the complex numbers as the so-called *Gaussian integers* $\mathbb{Z}[i]$. This makes E_8 into a 4-dimensional lattice over the Gaussian integers. It then consists of all vectors $(x_1, x_2, x_3, x_4) \in \mathbb{Z}[i]^4$ which satisfy

- $x_s - x_t \in (1+i)\mathbb{Z}[i]$;
- $\sum_{t=1}^{4} x_t \in 2\mathbb{Z}[i]$.

The 240 roots are the minimal norm vectors of the lattice. There is a symmetry group $(2\times 4^3)S_4$ generated by the diagonal matrix $\mathrm{diag}(i,i,1,1)$ and the coordinate permutations, under which the roots fall into three orbits, as follows:

- 16 roots of shape $(2,0,0,0)$;
- 96 roots of shape $(1+i,1+i,0,0)$;
- 128 roots of shape $(1,1,1,1)$.

To make this into E_8, we simply define the Euclidean norm to be half of the Hermitian norm on the ambient complex vector space.

3.3 An Eisenstein version of E_8

In the complex numbers, take $\omega = e^{2\pi i/3}$ and $\theta = \omega - \overline{\omega} = \sqrt{-3}$. Define a lattice by the following conditions on vectors (x_0,x_1,x_2,x_3):

- $x_t \in \mathbb{Z}[\omega]$;
- $x_1 + x_2 + x_3 \in \theta\mathbb{Z}[\omega]$;
- $x_0 + x_1 - x_2 \in \theta\mathbb{Z}[\omega]$.

Then we see a symmetry cycling (x_1,x_2,x_3), and another cycling $(x_0,x_1,-x_2)$, which together generate a group $2{}^{\cdot}A_4$ acting monomially on the four coordinates. Together with the diagonal element $\mathrm{diag}(\omega,1,1,1)$ this generates a symmetry group $3^4{:}2{}^{\cdot}A_4$. The roots (vectors of minimal norm in the lattice) fall into two orbits under this monomial group, as follows:

- 24 roots of shape $(\theta,0,0,0)$;
- 216 roots of shape $(0,1,1,1)$.

The Euclidean norm which makes this into E_8 is $2/3$ of the natural Hermitian norm.

The sublattice consisting of the vectors with $x_0 = 0$ is a copy of the E_6 lattice, and the sublattice of that which consists of vectors with $x_2 = x_3$ is a copy of the F_4 lattice.

3.4 A Hurwitz version of E_8

In this construction we take two copies of the F_4 lattice (which is the same as the D_4 lattice), identified with suitable scaled copies of the Hurwitz integral quaternions. In the quaternions, take the additive group generated by the D_4 lattice as described above. This gives us the ring $\mathbb{Z}[i,\omega]$ of *Hurwitz integral quaternions*. The units are the roots $\pm1,\pm i,\pm j,\pm k,\frac{1}{2}(\pm1\pm i\pm j\pm k)$ of D_4, and the elements of norm 2 are $(1+i)$ times these, that is $\pm1\pm i$, $\pm1\pm j$, $\pm1\pm k$, $\pm i\pm j$, $\pm i\pm k$, $\pm j\pm k$, which can be thought of as the long roots of F_4.

We now take 48 roots $(2r,0)$ where r is a short root of F_4, together with $24\times 8 = 192$ roots (r,qr) where r is a long root and $q \in Q_8$. If F_4 is labelled so that $\pm1\pm i$ are short and ±2, etc, are long, this gives

$(\pm2,\pm2,0,0 \mid 0,0,0,0)$	48
$(\pm2,0,0,0 \mid \pm2,0,0,0)$	64
$(\pm1,\pm1,\pm1,\pm1 \mid \pm1,\pm1,\pm1,\pm1)$	128

where in the last case there must be an even number of minus signs. Clearly we now see more symmetry, fusing the first two orbits of roots. We could also twist this by changing sign on one coordinate, so that there is an odd number of minus signs instead.

If instead F_4 is labelled so that $\pm 1, \pm i$ etc are short, and $\pm 1 \pm i$ etc are long, we get

$$\begin{array}{rl} (\pm 2, 0, 0, 0 \mid 0, 0, 0, 0) & 16 \\ (\pm 1, \pm 1, \pm 1, \pm 1 \mid 0, 0, 0, 0) & 32 \\ (\pm 1, \pm 1, 0, 0 \mid \pm 1, \pm 1, 0, 0) & 6.4.2.4 = 192 \end{array}$$

where in the last case the right hand pair of 1s can either be in the same positions as the left hand pair, or in the complementary positions. Labelling the coordinates $\infty, 0, 1, 3, 2, 6, 4, 5$ in that order enables us to describe the supports of the vectors of shape $(\pm 1^4, 0^4)$: they are either ∞ with a line $t, t+1, t+3 \pmod 7$ of the projective plane of order 2, or the complement thereof.

Now this construction of E_8 can be described as taking all vectors (x, y) with $x, y \in \mathbb{Z}[i, \omega]$ and $x + y \in (1 + i)\mathbb{Z}[i, \omega]$. In the left vector space $\mathbb{H}^2$ (with scalar multiplication given by $\lambda(x, y) = (\lambda x, \lambda y)$) there is now a monomial group of symmetries generated by right-multiplication by the diagonal matrices $\mathrm{diag}(\lambda, \mu)$ where $\lambda, \mu \in Q_8$, together with $\mathrm{diag}(\omega, \omega)$, and the coordinate permutation of order 2.

The roots fall into two orbits under this group, as follows:

- 48 roots of shape $(1 + i, 0)$;
- 192 roots of shape $(1, 1)$.

Alternatively, we may multiply by the scalar $1+i$ so that the roots become of shape $(2, 0)$ and $(1 + i, 1 + i)$.

3.5 An octonion version of E_8

We can identify the pairs of quaternions in the previous section with octonions in various ways, to obtain various octonion constructions of E_8. If we identify i, j, k with the octonions i_0, i_1, i_3 respectively, and then identify $(1 + i, 0)$ with $i_\infty = 1$ and $(0, 1 + i)$ with i_2 then we obtain

- 16 roots $\pm i_t$ coming from $(\pm 1 \pm i, 0), (\pm j \pm k, 0), (0, \pm 1 \pm i), (0, \pm j \pm k)$;
- 32 roots $\frac{1}{2}(\pm 1 \pm i_0 \pm i_1 \pm i_3)$ and $\frac{1}{2}(\pm i_2 \pm i_4 \pm i_5 \pm i_6)$ coming from the other roots which lie in just one coordinate;
- 64 similar roots with subscripts in one of the sets $\{\infty, 0, 2, 6\}$, $\{1, 3, 4, 5\}$, $\{\infty, 0, 4, 5\}$, $\{1, 2, 3, 6\}$, coming from the roots (x, y) with $x, y \in Q_8$.
- 64 roots with subscripts in one of $\{\infty, 1, 2, 4\}, \{\infty, 1, 5, 6\}$ or their complements, coming from $\omega(x, y)$;
- 64 roots with subscripts in one of $\{\infty, 2, 3, 5\}, \{\infty, 3, 4, 6\}$ or the complement, coming from $\overline{\omega}(x, y)$.

Now (*pace* Kirmse) this set of roots r is not closed under octonion multiplication, but it turns out that the set $\{(1 + i_0)r(1 + i_0)/2\}$ is closed under multiplication. It is known as the Dickson–Coxeter non-associative ring of integral octonions.

One way to obtain this multiplicatively closed version directly is to change the roots of the Hurwitz version of E_8, by replacing the diagonal symmetry $\mathrm{diag}(\omega, \omega)$ by $\mathrm{diag}(\omega, \overline{\omega})$, so that the roots are the images under the new monomial group of $(1 + i, 0)$ and $(1, 1)$. Or, in the scaled version, images of $(2, 0)$ and $(1 + i, 1 + i)$.

3.6 An icosian construction of E_8

From H_4: define $N : \mathbb{Q}[\sqrt{5}] \to \mathbb{Q}$ by $N(a + b\sigma) = a$, and define a new norm $N(q\overline{q})$ on H_4. Now the things of norm 1 are the original things of norm 1, together with their multiples by σ. Thus we obtain 240 roots.

(If we apply this process to H_2 we get A_4, and if we apply it to H_3 we get D_6.)

3.7 Another complex version of E_8

Consider the ring $\mathbb{Z}[\alpha]$ of complex numbers, where $\alpha = \frac{1}{2}(-1 + \sqrt{-7})$, so that $\alpha^2 + \alpha + 2 = 0$. Let $\beta = \overline{\alpha}$. We use a symmetry group $2{}^{\cdot}A_4$ acting on vectors (x_0, x_1, x_2, x_3), generated by (x_1, x_2, x_3) and $(x_0, -x_1, x_2)$. The 240 roots are then the images under this group of the following, where $x, y, z \in \{\alpha, \beta\}$:

- 8 roots of shape $(\alpha - \beta, 0, 0, 0)$;
- 24 roots of shape $(-xy, 1, 1, 1)$;
- 64 roots of shape $(1, x, y, z)$;
- 144 roots of shape $(0, 1, -x, yz)$.

(Note that there is a sign error in the description of this lattice on page 10 of the Atlas of Finite Groups [1].)

The automorphism group of this lattice is a double cover $2{}^{\cdot}A_7$ of the alternating group A_7. It may be generated by the monomial group $2{}^{\cdot}A_4$ together with the matrix

$$\frac{1}{\alpha - \beta}\begin{pmatrix} \beta - \alpha & 0 & 0 & 0 \\ 0 & -\beta & \beta^2 & 1 \\ 0 & \beta^2 & 1 & -\alpha \\ 0 & 1 & -\alpha & 2 \end{pmatrix},$$

which has order 7 and is normalized by the element of order 3 which cycles the last three coordinates.

References

[1] J. H. Conway, R. T. Curtis, S. P. Norton, R. A. Parker and R. A. Wilson, An ATLAS of Finite Groups, Oxford University Press, 1985. Reprinted with corrections, December 2003.

[2] H. S. M. Coxeter, Discrete groups generated by reflections, *Ann. Math.* **35** (1934), 588–621.