Anatomy of the Monster: II

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published in Proc. London Math. Soc. 84 (2002), 581–598

Abstract

We describe the current state of progress on the maximal subgroup problem for the Monster sporadic simple group. Any unknown maximal subgroup is an almost simple group whose socle is in one of 19 specified isomorphism classes.

1 Introduction

The Monster group $\mathbb{M}$ is the largest of the 26 sporadic simple groups, and has order

$$808\,017\,424\,794\,512\,875\,886\,459\,904\,961\,710\,757\,005\,754\,368\,000\,000\,000.$$

It was first constructed by Griess [3], as a group of 196884×196884 matrices. This construction was carried out entirely by hand.

In [9] Linton, Parker, Walsh and the second author constructed the 196882-dimensional irreducible representation of the Monster over $GF(2)$. In [4] Holmes and the second author constructed the 196882-dimensional representation over $GF(3)$.

Much work has been done on the subgroup structure of the Monster. Unpublished work by the first author includes the p-local analysis for $p \geq 5$, which was repeated (with corrections) by the second author, and extended to $p \geq 3$ (see [20]). The local analysis was completed with Meierfrankenfeld and Shpectorov's solution of the 2-local problem, also unpublished after several years [10]. The first author has also worked extensively on the non-local subgroups [11]. This includes the (relatively straightforward) case where the socle is non-simple, and more significantly the case where the socle contains an A_5 with $5A$-elements. Together with work on the $(2, 3, 7)$ structure constants, this reduced the number of isomorphism types of possible simple subgroups which need to be considered, from 55 to approximately 30. These results are summarised in [11], though in most cases the proofs are left as exercises for the reader.

In this paper, we reduce this number further to 19, and present some further restrictions on possible maximal subgroups. It is hoped that in due course the three recent computer constructions of the Monster [9], [4], [24] will lead to a complete determination of the maximal subgroups of the Monster, although this is still a long way off at present.

We take as our starting point the Atlas list of groups which are, or may be, involved in the Monster [2]. The list is reproduced in Table 1. For those 61 groups whose order divides that of the Monster and which were asserted in the Atlas not to be involved, proof of non-containment is provided in the last section of this paper. Note that the classification of local subgroups of $\mathbb{M}$ shows that none of these groups can be involved without being contained.

Six of the ten doubtful cases in Table 1 have now been resolved: $L_2(19)$, $L_2(29)$, $L_2(31)$, $L_2(49)$ and $L_2(59)$ are now known to be subgroups (see [7], [5], [22], [6]), and $L_2(41)$ and J_1 are known not to be involved (see [11], [18]).

The question of which of these groups are subgroups of $\mathbb{M}$ is in some cases rather more difficult to answer than the question of involvement. Our present state of knowledge is covered by Table 2. At this stage we cannot deal with the cases in (c), which seem to require deep analysis. We claim that the groups listed in (a) and (b) are subgroups of $\mathbb{M}$, while those in (d)

Table 1: The Atlas list of groups possibly involved in the Monster

$A_n, 5 \leq n \leq 12$
$L_2(q), q = 7, 8, 11, 13, 16, 17, 19?, 23, 25, 27?, 29?, 31?, 41?, 49?, 59?, 71?, 81$
$L_3(3), L_3(4), L_3(5), L_4(3), L_5(2)$
$U_3(3), U_3(4), U_3(5), U_3(8), U_4(2), U_4(3), U_5(2), U_6(2)$
$S_4(4), S_6(2), S_8(2), O_7(3), O_8^+(2), O_8^-(2), O_8^+(3), O_8^-(3), O_{10}^+(2), O_{10}^-(2)$
$Sz(8)?, G_2(3), G_2(4), {}^3D_4(2), F_4(2), {}^2F_4(2)', {}^2E_6(2)$
$M_{11}, M_{12}, M_{22}, M_{23}, M_{24}, Co_1, Co_2, Co_3, Fi_{22}, Fi_{23}, Fi_{24}', J_2, Suz$
$HS, McL, He, HN, Th, B, \mathbb{M}, J_1?$

Here a ? denotes a group for which it was not known at the time of publication whether or not it was involved in $\mathbb{M}$.

and (e) are not.

Theorem 1 *The groups listed in Tables 2(a) and (b), with the exception of $L_2(29)$ and $L_2(59)$, are subgroups of $\mathbb{M}$.*

Theorem 2 *The groups listed in Tables 2(d) and (e), with the exception of $L_3(5)$, are not subgroups of $\mathbb{M}$.*

Note. We deal with the case $L_3(5)$ in Section 2.4. The cases $L_2(29)$ and $L_2(59)$ have been dealt with computationally by Beth Holmes [5],[6].

Proof of Theorem 1. First, the known subgroups A_{12} and Th contain all the groups listed in the first rows of the two tables. Similarly, the subgroup Fi_{23} contains all the groups in the second and third rows of Table 2(a) plus $L_2(17)$, $L_2(16)$ and $U_4(2)$. The Harada–Norton group contains $U_3(5)$, and both the Held group and $O_{10}^-(2)$ are well-known subgroups of $3{\cdot}Fi_{24}'$, and therefore of $\mathbb{M}$. The subgroups $L_2(31)$ and $L_2(49)$ were found computationally, as subgroups of the Baby Monster [22], while $L_2(23)$ was found by Linton [8] as a subgroup of $3{\cdot}Fi_{24}'$. Finally, $U_3(4)$ is contained in $6{\cdot}Suz$. □

Before dealing with Theorem 2 we prove the following crucial result (proved by the first author many years ago, but not published at that time), which will also be useful later:

Table 2: The current status

(a) Completely classified subgroups of $\mathbb{M}$
$A_5, A_7, A_8, M_{12}, A_9, A_{10}, G_2(3), A_{11}, {}^3D_4(2), A_{12}, Th$
$L_2(25), S_4(4), S_6(2), L_4(3), U_5(2), {}^2F_4(2)', O_8^+(2), O_8^-(2), O_7(3)$
$S_8(2), O_8^+(3), Fi_{23}$
$L_2(23), L_2(29), L_2(49), L_2(59), U_3(5), He, O_{10}^-(2), HN$

(b) Partially classified subgroups of $\mathbb{M}$
$L_2(7), A_6, L_2(8), L_2(11), L_2(13), L_2(19), L_3(3), U_3(3), M_{11}, U_3(8)$
$L_2(17), L_2(16), L_2(31), U_4(2), U_3(4)$

(c) Doubtful cases
$L_2(27), L_3(4), Sz(8), L_2(71)$

(d) Involved but not contained
$M_{22}, U_4(3), L_5(2), M_{23}, HS, M_{24}, McL, U_6(2), Co_3, O_8^-(3), O_{10}^+(2)$
$Co_2, Fi_{22}, F_4(2), Co_1, {}^2E_6(2), Fi_{24}', B$
$L_2(81), L_3(5), J_2, G_2(4), Suz$

(e) Not involved in $\mathbb{M}$
$L_2(41), J_1$

Lemma 3 *There is no A_7 containing $5B$-elements in $\mathbb{M}$.*

Proof. Note first that there is only one class of $5B$-type A_5 which extends to S_5, namely those with type $(2B, 3B, 5B)$ and normalizer $S_5 \times S_3$. This is because the normalizers of the other two types are $(D_{10} \times A_5)^{\cdot}2$ and $A_5{:}4$. The latter group can be seen inside some of the direct product normalizers described in Table 4 of [11]. Indeed the existence of an S_5 or $A_5{:}4$ was used as a way of assigning some of the diagonal A_5s to classes T and B respectively in that table.

Suppose now that there is an A_7 containing $5B$-elements in $\mathbb{M}$. Then there is an S_5, containing 3-cycles of the A_7, so the 5-point A_5 is of $\mathbb{M}$-type $(2B, 3B, 5B)$. Now we look in the $3B$-centralizer $3^{1+12}{}^{\cdot}2^{\cdot}Suz$ for the subgroup $3 \times A_4$. There are two classes of involutions in $2^{\cdot}Suz$ which correspond to class $2B$ in the Monster, namely the central involution $-1A$, and the class $+2A$. In order to have a group $3 \times A_4$, we must obviously have the latter class. Moreover, the only 3-elements in $6^{\cdot}Suz$ which normalize a 2^2 group of type $+2A$ project to Suz-class $3C$. (To see this, observe that the 2^2-group decomposes the 12-space on which the group acts into a direct sum of three 4-spaces, and the normalizing element must permute these 4-spaces, so have trace 0, so be in class $3C$.) But by [20], these elements fuse to $\mathbb{M}$-class $3C$, contradicting the fact that they fuse to $\mathbb{M}$-class $3B$ via the S_5. □

Proof of Theorem 2. Of the cases in Table 2(e), $L_2(41)$ has been dealt with by the first author [11], and J_1 by the second author [18].

To deal with the first two rows of Table 2(d), we note that each of the groups therein has A_7 as a subgroup. Therefore, by Lemma 3, every subgroup of $\mathbb{M}$ of one of these types must contain $5A$-elements. But none of these groups are in Table 5 of [11], so none of them can be subgroups of $\mathbb{M}$. (Note: while it may not be clear in every case which class is being used in the argument which is suppressed in [11], all cases either have only one 5-class or contain a subgroup with this property which is also in Table 2(d).)

A similar argument covers $G_2(4)$, J_2, and Suz. These groups contain $A_5 \times A_4$, in which the A_5 must have $5A$-elements (as by Table 3 of [11] the centralizer of any $5B$-type A_5 is one of D_{10}, S_3 or 2). As the 5-elements in such A_5s centralize other A_5s containing the other conjugacy class of 5-element, it can be seen that that class too must fuse to $5A$ in the Monster.

Finally, $L_2(81)$ can be dealt with by means of its subgroup D_{80}, as no 40-element of $\mathbb{M}$ powering to $5B$ is conjugate to its inverse. □

2 Classification of subgroups

2.1 Elementary reductions

We now turn to the groups in Tables 2(a) and (b). These tables are distinguished by the fact that subgroups of $\mathbb{M}$ whose shape is one of the groups of Table 2(a) have been completely classified, but this is not true of Table 2(b). In this section we begin to justify our assertion that the 30 groups in Table 2(a) are completely determined.

By Lemma 3, any subgroup of $\mathbb{M}$ containing A_7 has $5A$-elements and therefore any occurrence must correspond to an item in the list in [11]. This deals with the cases A_7, A_8, A_9, A_{10}, A_{11}, A_{12}, $U_3(5)$, $S_6(2)$, $S_8(2)$, $O_7(3)$, $O_8^+(2)$, $O_8^-(2)$, $O_8^+(3)$, $O_{10}^-(2)$, Fi_{23}, and HN. It is also easy to deal with $L_4(3)$, $S_4(4)$ and He by showing that every subgroup of one of these shapes must also have a $5A$-element. Note that in none of these cases can there be any problem about which 5-class is being used, as either there is just one 5-class or the subgroup shown in the second column of Table 5 of [11] has already been completely classified.

The cases A_5 and $L_2(49)$ are also dealt with in [11], so, with the doubtful cases, as well as the postponed cases $L_2(29)$, $L_2(59)$ and $L_3(5)$, there remain 30 groups to consider, namely:

$$
\begin{array}{l}
L_2(q), q = 7, 8, 11, 13, 16, 17, 19, 23, 25, 27, 29, 31, 59, 71 \\
L_3(3), L_3(4), L_3(5), U_3(3), U_3(4), U_3(8), U_4(2), U_5(2) \\
G_2(3), {}^3D_4(2), {}^2F_4(2)', Sz(8), A_6, M_{11}, M_{12}, Th
\end{array}
$$

In every case where there is an A_5 subgroup we may assume that the elements of order 5 in that subgroup are in class $5B$, since the $5A$-cases have all been dealt with by the first author [11]. In fact, as in the cases considered above, [11] does not specify which class(es) of 5-elements in the subgroup are assumed to be in $\mathbb{M}$-class $5A$, so we need to be careful in some cases to ensure that we have proved that the particular 5-class we are using fuses to $5B$. This however only applies to the cases $L_3(5)$ and $U_3(4)$, which are considered individually below.

We now deal with six of the above 30 cases, namely, $U_5(2)$, M_{12}, $L_3(5)$, ${}^3D_4(2)$, Th, and $G_2(3)$.

2.2 $U_5(2)$

Any subgroup $U_5(2)$ contains $3 \times U_4(2)$, and by [11] we may assume that the elements of order 5 (which are all conjugate in $U_5(2)$) fuse to class $5B$. The $\mathbb{M}$-classes of elements of order 3 which commute with a $5B$-element are $3B$ and $3C$. But $C(3C) \cong 3 \times Th$ does not contain $3 \times U_4(2)$, so this case does not occur. In the other case, we have $C(3B) \cong 3^{1+12} \cdot 2 \cdot Suz$, and the elements of $\mathbb{M}$-class $5B$ project to Suz-class $5A$. But it is easy to see by character restriction that $2 \cdot Suz$ does not contain a subgroup $U_4(2)$ with $5A$-elements. Thus we have proved:

Theorem 4 *The Monster does not contain a subgroup $U_5(2)$ with $5B$-elements.*

2.3 M_{12}

The group M_{12} has a subgroup $2 \times S_5$. Now, as we saw in Lemma 3, only one of the three classes of $5B$-type A_5s in $\mathbb{M}$ extends to S_5, namely the one with centralizer S_3. In particular, the central involution in $2 \times S_5$ is in $\mathbb{M}$-class $2A$. But these involutions are inside A_5s in M_{12}, contradicting the fact that the Monster does not contain an A_5 of type $(2A, 5B)$. Thus we have proved:

Theorem 5 *The Monster does not contain a subgroup M_{12} with $5B$-elements.*

2.4 $L_3(5)$

In $L_3(5)$ there are two classes of 5-elements. The centralizer of an element of $L_3(5)$-class $5A$ contains 5^{1+2}, and therefore this class fuses to $5B$ in $\mathbb{M}$. By [11], we may assume that the other 5-class fuses to $5B$ as well. Moreover, there is a subgroup S_5, which must therefore be inside $S_3 \times Th$ in $\mathbb{M}$. In particular, the 3-elements are in class $3B$.

Now consider the subgroups $5^2{:}GL_2(5)$. By the 5-local analysis [20] we know that these must be either in

$$N(5B^2) \cong 5^{2+2+4}(S_3 \times GL_2(5))$$

or in

$$N(5^4) \cong 5^4{:}(3 \times 2^{\cdot}L_2(25)).2.$$

In the latter case, the subgroup $3 \times 2^{\cdot}L_2(25)$ is contained in $6^{\cdot}Suz$, and the non-central elements of order 3 project to class $3C$ in Suz, and therefore to $3C$ in $\mathbb{M}$, which is a contradiction.

We next investigate the structure of $5^{2+2+4}{:}(S_3 \times GL_2(5))$. The complement $S_3 \times GL_2(5)$ contains two distinct classes of $GL_2(5)$, one of which centralizes S_3, while the other centralizes just 2. In both cases, however, the centralizer of the group $5^2{:}GL_2(5)$ contains at least one $2A$-element. (This follows from the fact that the 3-normalizer $S_3 \times Th$ contains a subgroup $S_3 \times 5^2{:}GL_2(5)$ of this group.)

If we now consider generating $L_3(5)$ with two subgroups $5^2{:}GL_2(5)$ intersecting in $5^2{:}(4 \times 5{:}4)$, we see that all three subgroups are centralized by the same element of order 2. Therefore, whatever group is so generated is a subgroup of the Baby Monster. But in [19] the second author proved that $L_3(5)$ is not a subgroup of B, and so this does not happen.

Combining this with the result [11] that there is no $5A$-type $L_3(5)$ in $\mathbb{M}$, we have:

Theorem 6 *There is no subgroup $L_3(5)$ in the Monster.*

2.5 ${}^3D_4(2)$

The centralizer in ${}^3D_4(2)$ of a $7A/B/C$-element is $7 \times L_2(7)$, so these classes fuse to $\mathbb{M}$-class $7A$, because $C_{\mathbb{M}}(7B) \cong 7^{1+4}{:}2^{\cdot}A_7$, which does not contain a subgroup $L_2(7)$. If the $7D$-elements of ${}^3D_4(2)$ also fuse to $\mathbb{M}$-class $7A$, then this case is dealt with in [11], by accounting for the relevant $(2, 3, 7)$ structure constants. It turns out that there is a unique class of such subgroups ${}^3D_4(2)$ in $\mathbb{M}$, each with normalizer $S_4 \times {}^3D_4(2){:}3$.

So we may assume that the elements of ${}^3D_4(2)$-class $7D$ fuse to $\mathbb{M}$-class $7B$. Therefore the Sylow 7-subgroup of any such ${}^3D_4(2)$ has equal numbers of $7A$ and $7B$-elements. It follows from the class fusion given in [20] that the subgroup $7 \times L_2(7)$ is embedded in $C(7A) \cong 7 \times He$ in such a way that the $L_2(7)$ contains elements of He-class $7D/E$.

Now inspection of the list of maximal subgroups of He in [2] reveals that all the $L_2(7)$s of He-type $7D/E$ are contained in the involution centralizer $2^{1+6}L_3(2)$. In particular, the group $7 \times L_2(7)$ that we want is contained in the involution centralizer $2^{1+24}{}^{\cdot}Co_1$ in $\mathbb{M}$.

To generate ${}^3D_4(2)$, we can take a subgroup $7 \times 7{:}3$ of $7 \times L_2(7)$, and extend it to $7^2{:}2{\cdot}A_4$. But the full $\mathbb{M}$-normalizer of the 7^2-subgroup is $(7^2 \times D_{14}).(3 \times 2{\cdot}A_4)$, so there is a unique way to make this extension. Moreover, the extension is already visible in the centralizer of the same involution which centralizes $7 \times L_2(7)$. It follows that the group so generated is contained in $2^{1+24}{\cdot}Co_1$, and therefore is not ${}^3D_4(2)$.

Thus we have proved:

Theorem 7 *There is a unique class of groups of shape ${}^3D_4(2)$ in the Monster, and the normalizer of any such group is a non-maximal subgroup $S_4 \times {}^3D_4(2){:}3$.*

2.6 The Thompson group

From the above, the containment of ${}^3D_4(2)$ in Th shows that the 7-elements must fuse to $\mathbb{M}$-class $7A$. Therefore this case has been done in [11], since Th is a $(2,3,7)$-group. There is a unique class of Th in $\mathbb{M}$, normalized by the well-known maximal subgroup $S_3 \times Th$.

2.7 $G_2(3)$

Theorem 8 *Every $G_2(3)$ in $\mathbb{M}$ contains $7A$-elements, and has non-trivial centralizer.*

Proof. We show first that there is no $G_2(3)$ containing $7B$-elements in $\mathbb{M}$. So suppose for a contradiction that H is a subgroup $G_2(3)$ containing $7B$-elements. In [18] the second author showed that the only elements of order 3 which conjugate a $7B$-element to its square are in class $3C$. Hence the $3E$-elements in H fuse to class $3C$ in $\mathbb{M}$. On the other hand, there are pure 3^2 groups in H of $G_2(3)$-class $3A$, and also of $G_2(3)$-class $3B$. Thus neither of these classes can fuse to $3C$ in $\mathbb{M}$. But now the involution centralizer in $G_2(3)$ contains a 3^2-group containing elements of $G_2(3)$-classes $3A$, $3B$, $3D$ and $3E$, so at most two cyclic subgroups of this 3^2 can contain elements of $\mathbb{M}$-class $3C$. This is a contradiction, as there is no such 3^2 in $\mathbb{M}$.

Thus we have proved that any $G_2(3)$ in $\mathbb{M}$ contains $7A$-elements. The analysis of the $(2,3,7A)$-structure constants [11] shows that the subgroup $L_2(13)$ of our putative $G_2(3)$ is uniquely determined, up to conjugacy. It has centralizer $3^{1+2}{:}2^2$ of order 108, and double centralizer $G_2(3)$. Now *any* $G_2(3)$

containing this $L_2(13)$ can be generated by extending the 7-normalizer from D_{14} to 7:6. The normalizer of the D_{14} can easily be computed, by looking at the centralizers of involutory outer automorphisms of the Held group, and it has the shape $7{:}6 \times 3^{\cdot}S_7$.

Now the different ways of extending D_{14} to 7:6 fall into 4 orbits, of sizes $1+2+210+840$, under the action of $3^{\cdot}S_7$. In every case, however, the normal 3-group in $3^{\cdot}S_7$ actually centralizes both the $L_2(13)$ (since it is in $3^{1+2}{:}2^2$, which contains a Sylow 3-subgroup of $3^{\cdot}S_7$), and the 7:6, and therefore centralizes the group generated. It follows that every $G_2(3)$ in $\mathbb{M}$ has non-trivial centralizer, and hence its normalizer is not maximal. □

3 Computational results

3.1 Overview

There remain 24 cases to deal with, namely

$$L_2(q), q = 7, 8, 9, 11, 13, 16, 17, 19, 23, 25, 27, 29, 31, 59, 71$$
$$L_3(3), L_3(4), U_3(3), U_3(4), U_3(8), U_4(2), Sz(8), {}^2F_4(2)', M_{11}$$

In the following subsection we classify subgroups of types $L_2(25)$ and ${}^2F_4(2)'$. This uses substantial computations in subgroups of $\mathbb{M}$, but does not use any of the computer constructions of $\mathbb{M}$ directly.

A student of the second author, Beth Holmes, is working on classifying certain maximal subgroups computationally. She has already classified the subgroups $L_2(23)$, $L_2(29)$ and $L_2(59)$, and found that in each case there is a unique class, with normalizers $S_3 \times L_2(23)$, $L_2(29){:}2$ and $L_2(59)$, the first contained in $3^{\cdot}Fi_{24}$, the other two being new maximal subgroups [5],[6]. She has also eliminated the possibility of a subgroup $L_2(13)$ containing $13B$-elements, and is working on other cases.

We believe we can deal similarly with most of the remaining cases containing A_5, specifically

$$L_2(q), q = 9, 11, 16, 19, 31, 71, L_3(4) \text{ and } M_{11}$$

The plan is to find each A_5 in such a way that we can explicitly compute the 5-normalizer, and then run through all possible cases in the usual way, using an amalgamation of suitable subgroups.

3.2 $L_2(25)$ and ${}^2F_4(2)'$

In this subsection we aim to show that there is no $L_2(25)$ of $5B$-type in the Monster, and as a corollary, there is no ${}^2F_4(2)'$ of $5B$-type either. These results rely to some extent on computational results on subgroups of the Baby Monster [23], as well as other computations in smaller groups. We use the fact that $L_2(25)$ may be generated by subgroups S_5 and $5^2{:}4$ intersecting in 5:4. Thus we start by taking representatives of the two classes of $5B$-type S_5s, and trying to extend a subgroup 5:4 to $5^2{:}4$ inside the full $5B$-normalizer $5^{1+6}{:}2{}^{\cdot}J_2{:}4$. We shall show that there is a very small number of possible extensions, and in each case the group generated by the S_5 and the $5^2{:}4$ is centralized by an involution.

As usual, by [11] we may assume that all elements of order 5 in our putative $L_2(25)$ are in $\mathbb{M}$-class $5B$. Moreover, the existence of a subgroup S_5 implies (as in Lemma 3) that the 3-elements are in $\mathbb{M}$-class $3B$, and that the $\mathbb{M}$-centralizer of the A_5 in such an S_5 is a $(2A, 3C)$-type S_3. It follows that there are two possibilities for such an S_5 in $\mathbb{M}$. The first has centralizer S_3, and is a subgroup of the Thompson group Th, while the second has centralizer of order 2, generated by a $2A$-element. Thus we have proved:

Lemma 9 *There are exactly two classes of S_5 containing $5B$-elements in $\mathbb{M}$. The respective S_5-normalizers are $S_5 \times S_3$ and $S_5 \times 2$.*

Next we turn attention to the subgroup $5^2{:}12$ of our putative $L_2(25)$. We first prove from the 5-local analysis that there is only one class of 5^2 which is normalized by a $3B$-element in $\mathbb{M}$. For if the 5^2 is of type $5B_6(ii)$ or $5B_6(iii)$ in the notation of [20], then it is in the normal 5^4 of $5^4{:}(3 \times 2{}^{\cdot}L_2(25)).2$. But the only $3B$-elements in here are the ones centralizing $2{}^{\cdot}L_2(25)$, and these only normalize 5^2 groups which are 1-spaces over $GF(25)$—and the latter are of type $5B_6(i)$. Thus our 5^2 is of the type labelled $5B_6(i)$ in [20]. This means it is in the normal 5^{1+6} of the 5-normalizer. The normalizer of such a 5^2 in M is $5^{2+2+4}{:}(S_3 \times GL_2(5))$. Moreover, the only $3B$-elements in this group are the ones in a copy of $GL_2(5)$.

Next we determine the centralizers of the subgroups 5:4 which are contained in the two subgroups S_5. It can be seen that, as stated in Table 3 of [11], the centralizer of the D_{10} is $5^3{:}(4 \times A_5)$. This may be seen as the centralizer in $C(5B) \cong 5^{1+6}{:}2{}^{\cdot}J_2$ of the product of a $2B$-element of $2{}^{\cdot}J_2$ and a central 4-element of $4{}^{\cdot}J_2$. To extend this D_{10} to 5:4, we must adjoin an element mapping to $J_2{:}2$-class $4C$. There are two essentially different ways of doing this, as these elements are not conjugate to their negatives in $4{\cdot}J_2{\cdot}2 \cong 2{\cdot}J_2{:}4$. In one case the centralizer of $5{:}4$ is $5^2{:}(4 \times S_3)$, while in the other case it is $5{:}4 \times S_3$. Our next problem is to determine which of these occurs in each of our subgroups S_5.

We consider an embedding of the subgroup $5{:}4 \times S_3$ of $S_5 \times S_3$ into the 5-normalizer $5^{1+6}{:}2{\cdot}J_2{:}4$. As above, we see that the 4-element acts on the 5^6-factor with eigenvalues $1, 1, 4, 2, 2, 3$ (or their inverses) in one case, or $1, 4, 4, 2, 3, 3$ (or their inverses) in the other. The S_3 centralizes the two 1-dimensional eigenspaces, and acts as the deleted permutation representation on each of the 2-dimensional eigenspaces. (The former follows from the fact that in $2{\cdot}J_2$ it is the involutions of character value $+2$ in the 6-dimensional (5-modular) representation that fuse to class $2A$ in $\mathbb{M}$.) In particular, if we multiply our 4-element by an involution in the S_3, we obtain a 4-element in the other conjugacy class in $2{\cdot}J_2{:}4$. This means:

Lemma 10 *The subgroups* $5{:}4$ *in two nonconjugate subgroups* S_5 *of* $5B$*-type, are themselves not conjugate in* $\mathbb{M}$*. The centralizers of these two groups* $5{:}4$ *are* $5^2{:}(4 \times S_3)$ *and* $5{:}4 \times S_3$*.*

Since both these centralizers contain $3C$-elements, we can conjugate the two groups $5{:}4$ into a copy of the Thompson group Th. Their centralizers in Th are then 4 and $5{:}4$. Our central problem now is to determine which of these two Frobenius groups in Th is the one contained in an S_5 in Th. This turns out to be a very subtle question, which was first solved by computer calculations.

We began with a copy of $5{:}4 < S_5 < Th$, obtained by using the words in the standard generators provided in [17]. We then found the centralizer in Th of a 4-element in $5{:}4$, by first finding the involution centralizer using Bray's method [1]. This 4-centralizer contains 96 cyclic groups of order 5, and we tested each of them to see if it centralized the $5{:}4$. As none of them does centralize the $5{:}4$, it follows that the Frobenius group we started with has centralizer 4 in Th, and therefore centralizer $5^2{:}(4 \times S_3)$ in $\mathbb{M}$.

Remark. As the computer calculation here is very sensitive to any error, and could easily produce a false negative result, we ran another similar program which found an element of order 5 centralized by the D_{10}, but inverted by the element of order 4. It follows that inside the 5-normalizer $5^{1+2}{:}2{\cdot}A_4{:}4$

our 4-element acts on the 5^2-factor with eigenvalues $4, 3$ (or their inverses), and not with eigenvalues $1, 2$ (or their inverses).

We also give here a non-computational proof using the Y-diagram ([12], also Figure 1 of page 233 of the ATLAS [2]).

Lemma 11 *The* $5{:}4$ *inside the* S_5 *inside* Th *has centralizer* $5^2{:}(4 \times S_3)$ *in* $\mathbb{M}$*. The* $5{:}4$ *inside an* S_5 *of the other type has centralizer* $5{:}4 \times S_3$ *in* $\mathbb{M}$*.*

Proof. Using the notation of [12] and [2], we adjoin to the generators shown an involution Δ which normalizes each factor of the group $S_6 \times S_6 \times S_6$ generated by all the nodes except a. This can be chosen to centralize $c_i d_i e_i f_i$, $1 \leq i \leq 3$, and looking inside $\langle \Delta, a, b_1, c_1, d_1, e_1, f_1 \rangle \cong U_3(5){:}2$ shows that Δa has order 5.

The centralizer in Y_{555} of the S_3 which permutes the suffices is $Th \wr 2$. This has a subgroup $\langle ac_1c_2c_3, ad_1d_2d_3, ae_1e_2e_3, af_1f_2f_3 \rangle$, with the property that the subgroup S_5 of the Thompson group can be taken as its projection into one of the factors of the even part of Y_{555}. It can now be seen that the 4-element of the S_5 which normalizes the product of the $c_i d_i e_i f_i$s must involve a (since otherwise it would be odd in Y_{555}), and therefore it inverts Δa. This proves the last sentence of the above remark, from which the required result follows. □

We will be needing the following lemma later.

Lemma 12 *In each of the* $5B$*-type* S_5*s the* 4*-elements fuse to class* $4D$ *in* $\mathbb{M}$*.*

Proof. In the type of S_5 that centralizes an S_3, by the above argument we may take a 4-element to be the projection of $ad_1e_1f_1d_2e_2f_2d_3e_3f_3$ into one of the two factors of the even part of Y_{555}. In the notation of [12], this lies in the central product of the three groups $\langle d_i, e_i, f_i, z_i \rangle$ and the group $\langle a, b_1, b_2, b_3 \rangle$, where the common central involution is f^*. In our expression for the 4-element above, we may conjugate a to the involution corresponding to the extending node of the D_4-diagram whose nodes are a and the b_is, which is $b_1b_2b_3f^*$. We may also conjugate each of the $d_i e_i f_i$s to its product with f^*. This takes our 4-element to the product of the three $b_i d_i e_i f_i$s. This is a 4-element in the even part of $\langle b_1b_2b_3, c_1c_2c_3, d_1d_2d_3, e_1e_2e_3, f_1f_2f_3 \rangle$, which projects to a subgroup A_6 of the Thompson group. According to the ATLAS, such 4-elements belong to class $4B$ in the Thompson group, which fuses to $4J$ in the double cover of the Baby Monster and $4D$ in the Monster.

For the other type of S_5 the argument is easier, as by Lemma 11 the 4-element centralizes a 5-element, so it must belong to class $4B$ in Th and $4D$ in $\mathbb{M}$. □

We continue our investigation inside the 5-normalizer $5^{1+6}{:}2^{\cdot}J_2{:}4$. We now know exactly how our two Frobenius groups embed in here. For the sake of clarity, in each case choose the 4-element which squares the 5-elements in this Frobenius group. Then in the first case, where the S_5 has centralizer S_3, we know that the 4-element has eigenvalues $1, 1, 4, 2, 2, 3$, since for any automorphism of 5^{1+6} the determinant of its action on the central quotient is the cube of its eigenvalue on the centre. Similarly, in the second case the eigenvalues are $1, 4, 4, 2, 3, 3$.

In particular, in the second case, there is a unique extension of the 5 to a 5^2 on which the 4-element acts as a scalar. Also, this extension is centralized by the $2A$-element which centralizes the S_5. Therefore the group generated by S_5 and $5^2{:}4$ is also centralized by this involution.

In the first case, the eigenvalue 2 has multiplicity 2, so there are potentially six possible extensions. However, explicit calculation in the 6-dimensional representation of $2^{\cdot}J_2{:}4$ reveals that three of these contain $5A$-elements, so there are just three possible extensions, permuted by the S_3 which centralizes the S_5. Again, therefore, the resulting group is centralized by an involution. Thus:

Lemma 13 *Every $5B$-type $L_2(25)$ in M is centralized by a $2A$-element, so is contained in the Baby Monster.*

Theorem 14 *There is no $L_2(25)$ of $5B$-type in $\mathbb{M}$.*

Proof. According to the calculations in [23], there is no $5B$-type $L_2(25)$ in the Baby Monster. □

Corollary 15 *There is no ${}^2F_4(2)'$ of $5B$-type in M.*

Proof. The group ${}^2F_4(2)'$ contains $L_2(25)$. □

4 Partial results

In many cases partial classifications of certain types of subgroups are known. These will often be useful in limiting the amount of computation necessary to complete the classification. In this section we deal with all cases apart from groups of type $L_2(q)$, which will be covered in the next section.

4.1 $Sz(8)$

Our first result was proved by the second author in 1984, but the proof has not appeared in print before.

Theorem 16 *Any subgroup $Sz(8)$ in $\mathbb{M}$ contains $5B$-elements.*

Proof. Note first that $Sz(8)$ is a $(2,4,5)$-group in 6 independent ways. Also it contains a pure elementary Abelian 2^3, so the involutions must fuse to $\mathbb{M}$-class $2B$. Hence the 4-elements must fuse to $\mathbb{M}$-class $4A$, $4C$, or $4D$. Now using GAP [16] we calculate the relevant $(2,4,5A)$ structure constants in $\mathbb{M}$:

$$\begin{aligned}
\xi_{\mathbb{M}}(2B,4A,5A) &= \frac{40687}{14192640} < 1 \\
\xi_{\mathbb{M}}(2B,4C,5A) &= \frac{154601}{49152} = 3\frac{7145}{49152} < 4 \\
\xi_{\mathbb{M}}(2B,4D,5A) &= \frac{83}{160} < 1
\end{aligned}$$

so the only way an $Sz(8)$ of $5A$-type could exist (with trivial centralizer) is if it contains $4C$-elements. Moreover, since the structure constant is strictly less than 6, any $Sz(8)$ extends to $Sz(8){:}3$.

Now we look for the subgroup $3\times 5{:}4$ inside $N(5A) \cong (D_{10}\times HN){}^{\cdot}2$. The desired element of order 4 must correspond to an element of $HN{:}2$-class $2C$, $4D$, $4E$, or $4F$. These fuse respectively to elements of $2{}^{\cdot}B$-class $2C$, $4A$, $4F$, and $4J$, and therefore to elements of $\mathbb{M}$-class $4B$, $4A$, $4C$, $4D$ respectively. (These fusions are easily checked by character restriction.) Since our element is of $\mathbb{M}$-class $4C$, it is of $HN{:}2$-class $4E$. Hence the element of order 3 which commutes with it is in HN-class $3A$, and thence $2{}^{\cdot}B$-class $3A$ and $\mathbb{M}$-class

3A. It follows immediately from the power maps that the remaining outer classes of $Sz(8)$:3 fuse to $\mathbb{M}$-classes 6C, 12E, and 15A.

Finally, we observe that $Sz(8)$:3 can be generated by a triple of elements of $Sz(8)$-type $(2A, 3A, 15A)$, so of $\mathbb{M}$-type $(2B, 3A, 15A)$. But $\xi_{\mathbb{M}}(2B, 3A, 15A) = \frac{241}{6720} < 1$, so any such $Sz(8)$:3 has non-trivial centralizer. This contradiction completes the proof. □

4.2 $L_3(4)$

Theorem 17 *Every $L_3(4)$ in $\mathbb{M}$ is of type* $(2B, 3B, 4C, 4C, 4C, 5B, 7A)$.

Proof. We already know from [11] that the 5-elements are in class 5B, and therefore the involutions are in class 2B. Moreover, the Sylow 3-subgroup of $L_3(4)$ is a 3^2, while there is no 3C-pure 3^2 in $\mathbb{M}$, so the 3-elements are in class 3B. As noted in Section 2.7, it follows from [18] that the only elements of order 3 which conjugate a 7B-element to its square are in class 3C. Therefore the 7-elements are in class 7A.

Now according to [11] there is a unique class of $L_3(2)$s of type $(2B, 3B, 7A)$ in the Monster, centralizing S_4. To see such a group, we look inside $C_{\mathbb{M}}(S_4) = S_8(2)$. This group has a subgroup $O_8^-(2)$ in which $L_3(2)$ is maximal, and it can be seen that this $L_3(2)$ fuses to type $(2B, 3B, 4C, 7A)$ in $\mathbb{M}$. Hence all 4-elements in $L_3(4)$ would have to fuse to 4C-elements in $\mathbb{M}$. □

4.3 $U_3(3)$

Theorem 18 *Every $U_3(3)$ in $\mathbb{M}$ contains 2B-elements.*

Proof. If we have 2A-elements, then all 4-elements fuse to $\mathbb{M}$-class 4B. By looking at the unitary structure of $U_3(3)$, we can see that its Sylow 2-subgroup can be written in the form

$$\langle a, b, c | a^4 = b^4 = [a, b] = c^2 = 1, a^c = a^{-1}, b^c = ab \rangle.$$

However, such a group cannot occur in the Monster. To see this, we first note that b commutes with a. Now $A = C_{\mathbb{M}}(a)$ has structure $4{\cdot}F_4(2){\cdot}2$. (Note: the non-splitness of the outer extension can be seen by observing that class 2E of $F_4(2).2$ must lift to 8C in the Monster, which determines which group of the isoclinism class $4{}^{\cdot}F_4(2).2$ occurs.) As both c and ac fuse in $\mathbb{M}$ to $2A$, we have $C_{\mathbb{M}}(\langle a, c\rangle) = 2{}^{\cdot}F_4(2)$, a subgroup of index 4 in A, and the elements x of A can be split into four cosets according to which power of a is equal to $[x, c]$. In particular, since $[b, c] = a$, it follows that b lies in the outer half of $A \cong 4{}^{\cdot}F_4(2){}^{\cdot}2$. But there *are* no elements of order 4 in the outer half of A. This proves that $U_3(3)$s with $2A$-elements cannot occur in $\mathbb{M}$. □

If we have $7B$-elements then the class $3B$ in $U_3(3)$ must fuse to class $3C$ in $\mathbb{M}$, since these are the only 3-elements which properly normalize a $7B$-element.

If we now have an $L_2(7)$ of type $(2B, 3A, 7A)$, then by section 5 of [11] it has centralizer $2^2{}^{\cdot}L_3(4){:}S_3$. An A_4 inside it has centralizer $2^{11}{}^{\cdot}M_{24}$, and an S_4 has centralizer of order $2^{18}.3^3.5.7$, using the structure constants

$$\begin{aligned}
\xi_{\mathbb{M}}(2B, 3A, 4A) &= 1/2^{18}.3^3.5.7\\
\xi_{\mathbb{M}}(2B, 3A, 4C) &= 1/2^{14}.3.7\\
\xi_{\mathbb{M}}(2B, 3A, 4D) &= 0.
\end{aligned}$$

In particular the 4-elements are in class $4A$, and the centralizer of the S_4 is $2^{11}{}^{\cdot}L_3(4){:}S_3$. Again we see that two copies of $2^2{}^{\cdot}L_3(4){:}S_3$ in here intersect nontrivially, so any $U_3(3)$ of this type has non-trivial centralizer.

Next suppose we have an $L_2(7)$ of type $(2B, 3B, 7A)$. Then the same argument as for $L_3(4)$ shows that $4C$ in $U_3(3)$ fuses to $4C$ in $\mathbb{M}$. Finally, by [11] there is no $L_2(7)$ of type $(2B, 3C, 7A)$, so the only remaining possibilities are as in Table 3.

4.4 $U_4(2)$

For $U_4(2)$, note that the Baby Monster B does not contain an elementary abelian 2^4 which lifts to $Q_8 \circ Q_8$ in the double cover $2{}^{\cdot}B$, so the $2A$-elements fuse to $\mathbb{M}$-class $2B$. The subgroup S_6 implies that class $2B$ in $U_4(2)$ fuses to class $2B$ in $\mathbb{M}$, and $3C$ and $3D$ fuse to $3B$, and also (by Lemma 12) that $4B$ fuses to $4D$. The 9-elements imply that $3AB$ fuses to $3B$.

4.5 $U_3(8)$

For $U_3(8)$, the subgroup D_{18} implies that $2A$ fuses to $2B$ and $3C$ fuses to $3B$. Then the analysis of the $(2, 3, 7)$ structure constants in [11] implies that the subgroup $L_2(8)$ is of type $(2B, 3B, 7A)$ in $\mathbb{M}$. The elements of order 21 may then be either in class $21A$ or in class $21C$, with 7th powers respectively in $3A$ and $3C$.

In the latter case, note that $C_{\mathbb{M}}(3C) \cong 3 \times Th$ contains a unique class of $L_2(8)$, and such an $L_2(8)$ centralizes at least 3^2. Therefore from [11] its centralizer is $3^{\cdot}S_6$, and the whole normalizer $L_2(8){:}3 \times 3^{\cdot}S_6$ is a maximal subgroup of $N(3A) \cong 3^{\cdot}Fi_{24}$. In particular the elements of order 9 are 5th powers of elements of order 45, so in $\mathbb{M}$-class $9A$.

4.6 $L_3(3)$

First we show that a $13B$-element is normalized only by $3C$-elements, not $3A$ or $3B$, while a $13A$-element is normalized by $3B$-elements and $3C$-elements, but not $3A$.

A $13B$-element may be found inside $6^{\cdot}Suz$, where it is normalized by elements of Suz-class $3C$ only. These lift to elements of $\mathbb{M}$-class $3C$ only.

A $13A$-element has normalizer $(13{:}6 \times L_3(3))^{\cdot}2$, and a subgroup $6 \times L_3(3)$ thereof can be found in $6^{\cdot}Suz$. The 3-elements normalizing the $13A$-element are then either central in $6^{\cdot}Suz$, in which case they are in $\mathbb{M}$-class $3B$, or lift to class $3B$ or $3C$ in Suz, in which case they are in $\mathbb{M}$-class $3B$ or $3C$ by [20].

But now the structure constants $\xi_{\mathbb{M}}(2A, 3B, 13A) = 0$, $\xi_{\mathbb{M}}(2A, 3C, 13A) = 1/36$ and $\xi_{\mathbb{M}}(2A, 3C, 13B) = 0$ show that any $L_3(3)$ containing $2A$-elements has nontrivial centralizer. Therefore we may restrict attention to subgroups $L_3(3)$ containing $2B$-elements.

The $3A$-elements in $L_3(3)$ cannot fuse to $3C$ since they form pure 3^2-groups. This leaves just the cases listed in Table 3.

4.7 M_{11}

The fact that M_{11} contains S_5 tells us that the A_5 in any $5B$-type M_{11} is of type $(2B, 3B, 5B)$. Moreover, by Lemma 12 the 4-elements in such an S_5 are in class $4D$, and hence the 8-elements are in class $8F$.

4.8 $U_3(4)$

There is a unique class of subgroups A_5 in $U_3(4)$, and the existence of a subgroup $5 \times A_5$ of $U_3(4)$ implies that the A_5 is of type $(2B, 3C, 5B)$, and that the central 5-elements are also in class $5B$. The rest of the class fusion given in Table 3 follows from the power maps.

5 $L_2(q)$

5.1 General results

From Table 1, the set of values of $q > 5$ for which $L_2(q)$ might be a subgroup of the Monster is $\{7, 8, 9, 11, 13, 16, 17, 19, 23, 25, 27, 29, 31, 41, 49, 59, 71, 81\}$. Following Section 6 of [11], we write $Q = \{19, 27, 29, 31, 41, 49, 59, 71\}$. It is convenient to give a proof of the following result, which follows from what was stated in [11]. Note that this theorem does not cover the cases $q = 16$, 25 or 81.

Theorem 19 *If $q \in Q$ or $q = 7, 8, 9, 11, 13, 17, 23$, then the only possible subgroups of $\mathbb{M}$ of type $L_2(q)$ with $5A$-elements are those described in Table 5 of [11].*

Proof. If $q = 7$, 8, 13, 17, 23 or 27 then the theorem is obvious as there are no elements of order 5 at all. The other cases are those in which $L_2(q)$ is generated by two groups of type A_5 which may be chosen to intersect in either D_{10} or A_4, and this is the property we use.

There are five cases according to the conjugacy class fusion of the A_5. In the two cases where the 3-element belongs to class $3C$, the result is clear, as it follows from Table 3 of [11] that each A_4 extends uniquely to an A_5. We deal with the other three cases in turn.

If the A_5 is of type $(2A, 3A, 5A)$ then, from Table 3 of [11], the centralizer of the $L_2(q)$ is isomorphic to the intersection of a pair of distinct A_{12}s in HN, and also to the intersection of a pair of distinct A_{12}s in $O_{10}^-(2)$. The possibilities for both these intersections are described in the ATLAS (pages 147 and 166). The only possible intersections that occur in both cases are M_{12} and the even parts of $S_6 \wr 2$ and $S_3 \wr A_4$. The centralizers of these groups are $L_2(11)$, A_6 and $3^4{:}A_5$ respectively, and as each of these has A_5 as a maximal subgroup it follows that it is indeed generated by a pair of A_5s, and in the first two cases they can clearly be chosen to intersect in either D_{10} or A_4. This exhibits two of the cases in Table 5 of [11].

If the A_5 is of type $(2B, 3A, 5A)$ then, using the subgroup D_{10}, it follows from Table 3 of [11] that the centralizer of the $L_2(q)$ is isomorphic to the intersection of a pair of distinct subgroups $2.M_{22}.2$ in $2.HS.2$. This intersection can be either $2.M_{21}.2$ or $2^5.S_6$, which can be shown to centralize $2 \times A_6$ and $2^5.A_5$ respectively. Of these cases only the first can give rise to an $L_2(q)$, namely A_6. This case can also be seen in Table 5 of [11].

If the A_5 is of type $(2B, 3B, 5A)$ then, using the subgroup A_4, it follows from Table 3 of [11] that the centralizer of the $L_2(q)$ is isomorphic to the intersection of a pair of M_{11}s in $2.M_{12}.2$. (Note that as the closure of the A_5, namely $S_6.2$, contains more than one A_5, there is no requirement for the M_{11}s to be distinct.)

In the case when the M_{11}s are conjugate in $2.M_{12}$, their intersection is M_{11} or A_6. These groups can be shown to centralize $S_6.2$ and the even part of $S_6 \wr 2$ respectively. Only the first case can give rise to an $L_2(q)$, namely A_6. This case can also be seen in Table 5 of [11].

Finally, in the case when the M_{11}s are not conjugate in $2.M_{12}$, their intersection is $L_2(11)$, which centralizes M_{12}. The group generated by the two A_5s, which must centralize $L_2(11)$ and no more in the Monster, can only be the type of $L_2(11)$ which is non-transitive on the 12 points on which the M_{12} acts. □

5.2 $L_2(q)$ for $q \leq 17$

In this section we consider the possibilities for subgroups of type $L_2(q)$, where $q \in \{7, 8, 9, 11, 13, 16, 17\}$. The results of [11], which were proved in the previous subsection except when $q = 16$, show that the cases with $5A$-elements are known. The 6-transposition property, together with the fact that the product of two transpositions cannot belong to class $5B$, shows that unless $q = 7$ the involutions of $L_2(q)$ belong to $\mathbb{M}$-class $2B$. For $L_2(7)$, the $(2, 3, 7)$ structure constant results of [11], together with the fact that a $7B$-element is only properly normalized by a $3C$-element, show that any remaining $L_2(7)$ is of type $(2B, 3C, 7B)$.

For $L_2(9) \cong A_6$, the classification of A_5s, together with the fact that every 3^2 containing $3C$-elements contains exactly six $3C$-elements, shows that every $5B$-type A_6 has type $(2B, 3B, 3B, 5B)$.

For $L_2(11)$ and $L_2(16)$ the class fusions given in Table 3 follow immediately from the classification of A_5s, together with the power maps. The fact that all 9-elements cube to $3B$ gives the stated result for $L_2(17)$ as well.

For $L_2(8)$ and $L_2(13)$, both of which are $(2, 3, 7)$-groups, the structure constant analysis in [11] shows that they have type $(2B, 3B, 7B)$ or $(2B, 3C, 7B)$. Moreover, the latter case is impossible for $L_2(8)$, as the 3-elements are cubes. Finally, for $L_2(13)$, note that a $13B$-element is not properly normalized by a $3B$-element (see Section 4.6 above), and Holmes has eliminated the possibility of any other $L_2(13)$ with $13B$-elements [6].

5.3 $L_2(q)$ for $q \geq 19$

The cases $q = 25, 41, 49, 81$ have been dealt with, and the cases $q = 23, 29, 59$ have been completely classified by Holmes [5], [6]. However, we consider all $q \in Q$ (which includes the cases $q = 29, 41, 49, 59$ but not $q = 23, 25, 81$) as this enables us to fill in some of the details which were omitted from the published proofs in [11].

Theorem 20 *If $q \in Q$, then, in any subgroup of $\mathbb{M}$ of shape $L_2(q)$, the involutions must fuse to $2B$-elements, elements of order 5 (unless $q = 27$) to $5B$-elements, and elements of order 3 to $3B$-elements.*

Proof. As stated in [11], the involutions of any such $L_2(q)$ must belong to class $2B$ because the product of any two $2A$-elements has order at most 6. It follows from [11] and was proved above in Theorem 19 that the 5-elements (where they exist, *i.e.,* unless $q = 27$) belong to class $5B$. It remains to prove, as asserted in Section 6 of [11], that the 3-elements belong to class $3B$.

For $q = 27$ this follows from the fact that $\mathbb{M}$ has no $3A$- or $3C$-pure elementary abelian subgroups of order 27 (which can be seen from the fusion maps from $3{}^{\cdot}Fi_{24}'$ and $3 \times Th$ given in [20]). For $q = 19$ or 71, we use the fact that all 9-elements power to $3B$-elements.

The remaining cases are $q = 29, 31, 41, 49, 59$. Suppose, for a contradiction, that such a subgroup $L_2(q)$ contains elements of class $3A$ or $3C$. By Table 3 of [11], any A_5 containing $5B$-elements but not $3B$-elements is of type $(2B, 3C, 5B)$ and centralizes D_{10}, in which the 5-elements also belong to class $5B$ (as $5A$ cannot centralize $3C$).

There are three conjugacy classes of $2B$-pure four-groups $\langle t_1, t_2 \rangle$ in $\mathbb{M}$; they are distinguished by having composition factors M_{24}, M_{12} and A_8 in their centralizers. Only the third of these types centralizes a $5B$-element. Now, if we put $G_0 = C_{\mathbb{M}}(t_1) \cong 2^{1+24}{}^{\cdot}Co_1$ and $G_1 = G_0/O_2(G_0) \cong Co_1$, then the image of t_2 in G_1, in the three cases above, belongs to class $1A$,

$2C$ and $2A$ respectively. In particular, in the third case, which by the above argument must hold for any four-group in a $(2B, 3C, 5B)$-type A_5, the image of t_2 is a 6-transposition in G_1, *i.e.,* its product with any G_1-conjugate has order at most 6.

If we fix t_1 then we can choose a pair of involutions commuting with t_1 (so that they have all the properties of t_2) in such a way that their product, u, has respective order $7, 16, 5, 24, 15$ in G_0, according as $q = 29, 31, 41, 49, 59$. As the kernel of the map from G_0 to G_1, namely 2^{1+24}, has exponent 4, the first and last cases contradict the 6-transposition property. For $q = 31$, u^4 must be a 4-element of 2^{1+24}, which belongs to class $4A$ in $\mathbb{M}$; but it can be seen from the class list of $\mathbb{M}$ that no such element exists. For $q = 41$, u must belong to class $5B$ of $\mathbb{M}$, but any 5-element of G_1 which is the product of two $2A$-elements lifts in $\mathbb{M}$ to a $5A$. And for $q = 49$, u^8 must belong to class $3C$ (as by hypothesis the 3-elements in our $L_2(49)$ are $3C$s), but whenever two $2A$-elements of G_1 have product of order 6 the 3-part of this product must lift in $\mathbb{M}$ to a $3A$. □

We now deduce the class of the 7-elements of $L_2(q)$ when $q = 29, 49, 71$. If $q = 29$ then the 7-element can be seen inside $N(29A) = 29{:}42$, where the 42-element has p-parts $2B$ and $3A$, and hence $7B$. If $q = 49$ then the 7-elements must be $7A$s as only $3C$s can properly normalize $7B$s [18]. And if $q = 71$ then the 7-elements are the fifth powers of 35-elements whose 5-parts belong to class $5B$, so must belong to class $7B$.

By [11], if $q = 31$ or 71, the A_4 in the A_5 in any $L_2(q)$ has centralizer $2^{1+6}.3^{1+2}.4$, whereas an S_4 of type $(2B, 3B, 4A)$ has centralizer $2{}^{\cdot}M_{12}$. Therefore the S_4 in $L_2(q)$ cannot be of this type, so the 4-elements fuse to $4C$. If $q = 71$ then the 36-elements must fuse to $36D$, and the whole class fusion is determined.

It follows from the above that the class fusion map for $L_2(29)$ is

$$(2B, 3B, 5B, 7B, 14C, 15C, 29A),$$

and the computations in [6] show that the map for $L_2(59)$ is

$$(2B, 3B, 5B, 6E, 10E, 15C, 29A, 30G, 59AB).$$

We append in Table 3 a list of what is currently known about the possible class fusions from each of the remaining 19 groups. This enables us to prove:

Theorem 21 *Any proper subgroup of the Monster with $2A$-elements lies inside one of the known maximal subgroups.*

Note. The list of known maximal subgroups of the Monster consists of all the non-local subgroups shown on page 234 of the ATLAS, and the local subgroups with no supergroups shown, except that the last of the 5-locals has structure $5^4{:}(3 \times 2{\cdot}L_2(25)){:}2_2$ with the outer involution inverting the 3, and four extra subgroups should be added, a 7-local of shape $7^2{:}SL_2(7)$ where the O_7-subgroup is $7B$-pure, the 41-local 41:40 (because the supergroup $L_2(41)$ is now known not to exist), and the new subgroups $L_2(29){:}2$ and $L_2(59)$ [5],[6].

Proof. Any further maximal subgroup, apart from the odd-order group 71:35, must have as its socle a group of one of the types shown in Table 3, and thus lie between such a group and its automorphism group.

It is easy to see that any $2A$-type involution in such a group must lie in the outer half of a group $G.2$, where G is one of the groups shown in Table 3. Looking through all cases we find that in every case the product of any such involution and one of its conjugates can be chosen to have order at least 7, or to belong to class $3B$ or $5B$, which is impossible for a $2A$-type involution. □

6 The ATLAS results

We now redeem our pledge to prove that the simple groups stated in the ATLAS as not being involved in the Monster are indeed not contained. As we said earlier, the classification of maximal local subgroups shows that they cannot be involved without being contained.

For groups containing A_7, Lemma 3 shows that any 5-elements in the A_7s must be $5A$s, so that we can use the classification in [11]. This covers the following cases: A_n $(13 \le n \le 32)$, $L_n(4)$ $(4 \le n \le 6)$, $L_4(7)$, $L_6(3)$, $U_6(4)$, $S_4(7)$, $S_6(4)$, $S_{10}(2)$, $S_{12}(2)$, $O_7(5)$, $O_9(3)$, $O_{10}^+(3)$, $O_{12}^+(2)$, $O_{12}^-(2)$, $O'N$, Ru.

For most of the other groups we can exhibit a subgroup which is known not to be contained in $\mathbb{M}$: D_{62} in $L_2(32)$, D_{46} in $L_2(47)$, 63 in $L_2(64)$, 63 in $L_2(125)$, 1023 in $L_2(1024)$, 91 in $L_3(9)$, 91 in $L_2(16)$, 217 in $L_3(25)$, $L_3(5)$ in $L_4(5)$, 91 in $L_4(9)$, 121 in $L_5(3)$, 65 in $U_4(4)$, $3{\cdot}U_3(5)$ in $U_4(5)$, 63 in $U_4(8)$, 65 in $U_5(4)$, 63 in $S_4(8)$, $L_2(81)$ in $S_4(9)$, $L_3(5)$ in $S_6(5)$, D_{62} in $Sz(32)$, $L_3(5)$ in $G_2(5)$.

This leaves just the following cases: $L_3(7)$, $S_4(5)$, $S_6(3)$, J_3. The case J_3 was solved by Griess and Smith (Lemma 14.4 of [3]). Next, $S_4(5)$ can be eliminated by using [11] and our result (Theorem 14) that any $L_2(25)$ has $5A$-elements.

Table 3: Class fusions not yet eliminated

Group	Class fusions
$L_2(7)$	$2B, 3C, 4, 7B$
A_6	$2B, 3B, 3B, 4, 5B$
$L_2(8)$	$2B, 3B, 7B, 9$
$L_2(11)$	$2B, 3B/B/C, 5B, 6B/E/F, 11A$
$L_2(13)$	$2B, 3B/B/C, 6B/E/F, 7B, 13A$
$L_2(17)$	$2B, 3B, 4, 8, 9, 17A$
$L_2(19)$	$2B, 3B, 5B, 9, 19A$
$L_2(16)$	$2B, 3B/C, 5B, 15C/D, 17A$
$L_3(3)$	$2B, 3A/B/B, 3C, 4, 6C/B/E, 8, 13$
	$2B, 3B, 3B, 4, 6B/E, 8, 13A$
$U_3(3)$	$2B, 3A/B/B, 3B, 4, 4C, 6C/B/E, 7A, 8, 12$
	$2B, 3A/B/B, 3C, 4, 4, 6C/B/E, 7B, 8, 12$
M_{11}	$2B, 3B, 4D, 5B, 6B/E, 8F, 11A$
$L_2(27)$	$2B, 3B, 7B, 13, 14C$
$L_2(31)$	$2B, 3B, 4C, 5B, 8A/E, 15C, 16A/B, 31AB$
$L_3(4)$	$2B, 3B, 4C, 4C, 4C, 5B, 7A$
$U_4(2)$	$2B, 2B, 3B, 3B, 3B, 4, 4D, 5B, 6, 6, 6, 6, 9, 12$
$Sz(8)$	$2B, 4, 5B, 7, 13$
$U_3(4)$	$2B, 3C, 4, 5B, 5B, 10D/E, 13, 15D$
$L_2(71)$	$2B, 3B, 4C, 5B, 6E, 7B, 9B, 12I, 18D, 35B, 36D, 71AB$
$U_3(8)$	$2B, 3A/A/C, 3B, 4, 4, 4, 6C/C/F, 7A, 9A/B/A, 19A, 21A/A/C$

Note: Alternatives where given should be read in parallel. For example, an $L_2(11)$ is of type $(3B, 6B)$ or $(3B, 6E)$ or $(3C, 6F)$.

For $L_3(7)$, take a subgroup $7^2{:}2{}^{\cdot}L_2(7){:}2$ and call its O_7-subgroup G. If $L_3(7)$ were a subgroup of the Monster, then from [20] we know that G would have to fuse to a $7B$-pure group. But in $\mathbb{M}$ $7B$-elements are properly normalized only by $3C$-elements (see [18]), so the Sylow 3-subgroup of $L_3(7)$ would have to fuse to a $3C$-pure group of order 9, which is known to be impossible.

As for $S_6(3)$, the $3AB$-elements are cubes, so must fuse to $3B$-elements in $\mathbb{M}$. So their centralizers $3^{1+4}{:}2{}^{\cdot}U_4(2)$ must be contained in $3^{1+12}{\cdot}2{}^{\cdot}Suz$. But an easy character restriction of the 12-character of $6{}^{\cdot}Suz$ shows that this does not happen.

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