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ON THE SIMPLE GROUPS OF SUZUKI AND REE

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Contents

Abstract

We develop a new and uniform approach to the three families of twisted simple groups of Lie type discovered by Suzuki and Ree, without using Lie algebras. A novel type of algebraic structure is defined, whose automorphism groups are the groups in question. This leads to elementary proofs of the group orders and simplicity, as well as much information on subgroup structure and geometry.

1. *Introduction*

Around 1960 the last three infinite families of finite simple groups were discovered. These were the Suzuki groups, and two families of Ree groups. Suzuki [**15**] constructed his groups as groups of 4×4 matrices, over a field of characteristic 2 and odd degree. Ree's approach [**13**, **14**] was more abstract, and he constructed his groups as centralizers of certain outer automorphisms in Chevalley groups of type G_2 (in characteristic 3) and F_4 (in characteristic 2). The latter approach also yields the Suzuki groups when applied to Chevalley groups of type B_2. It also generalizes to perfect infinite fields with a so-called Tits automorphism, that is, one which squares to the Frobenius automorphism $x \mapsto x^p$, where p is the characteristic. Nevertheless, the machinery behind these constructions is formidable, as it involves first constructing the Lie algebras, then the Chevalley groups as groups of automorphisms of the algebras, and using much detailed structural information in order to construct the automorphisms of the groups, and the centralizers of the automorphisms. This 'standard approach' is well exposed in Carter's book [**2**], though not, unfortunately, in complete detail.

Tits [**17**, **18**] made some simplifications to the constructions by interpreting all these groups as groups of automorphisms of certain geometries. In the case of the Suzuki groups, the resulting 'ovoid' of q^2+1 points (where q is the order of the underlying field) was already implicit in Suzuki's work, and the group acts 2-transitively on the points of the ovoid. In the G_2 case, the so-called Ree–Tits unital has q^3+1 points, on which again the group acts doubly-transitively. In the F_4 case, the result is a 'generalized octagon', which contains $(q+1)(q^3+1)(q^6+1)$ points and $(q^2+1)(q^3+1)(q^6+1)$ lines. Each line contains $q+1$ points, and each point lies on q^2+1 lines. Nevertheless, the geometries could not really be constructed without at least

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some motivation from the Suzuki–Ree constructions, and the calculations required were still formidable, and consequently not published in full.

More recently, other approaches have been tried in order to simplify the constructions of these groups further. This is not really necessary in the case of the Suzuki groups, which are small enough that any number of elementary approaches will work. There are constructions for example in the books of Huppert and Blackburn [**9**], Taylor [**16**], and Geck [**7**] as well as Lüneburg [**11**] and van Maldeghem [**20**]. In the case of the 'small' Ree groups (those of type G_2), there is a recent paper by de Medts and Weiss [**6**] which fills in the details of the Tits construction.

The large Ree groups (those of type F_4) are however very much harder to construct. Tits [**18**] published a construction in 1983, and there is another in the book of Tits and Weiss [**19**] from 2002. Nevertheless, when I came to write about these groups for my book [**21**], I did not find anything at a suitably elementary level anywhere in the literature, so I set about re-constructing the groups for myself. The result of this work [**25**] appeared in 2010, and gives arguably the first genuinely elementary proof of existence of the large Ree groups. Remarkably, most of the geometrical part of this work had already been done, in a rather different way, by Coolsaet [**3**, **4**, **5**], although I was not aware of it at the time, and he was not trying to re-construct the groups, but rather to understand the generalized octagon.

In the course of this work, I explored a number of different approaches to the Suzuki groups [**22**] and small Ree groups [**23**, **24**] as well. By considering all three cases in parallel, I am now able to make significant further simplifications. In particular, the definitions of the bullet product (which I now rename the star product), the Weyl group and the root groups are better motivated and no longer appear so arbitrary, and most of the substantial calculation which was suppressed in [**25**] is now unnecessary. Moreover, I no longer rely on constructions of octonions or the exceptional Jordan algebra, but instead use directly the algebraic structure of the root lattices as rings of integral complex numbers or quaternions.

In this paper we develop this new theory of the Suzuki and Ree groups, proving everything from first principles. The groups are defined as automorphism groups of a new kind of algebraic structure, with three different products defined on it. This structure is defined in Section 3, using the rings of Gaussian, Eisenstein, and Hurwitz integers described in Section 2 as motivation. In Sections 4 and 6 we construct some automorphisms, which in the Lie theory are known as the Weyl group, the maximal split torus, and the root groups, but whose definitions come entirely from the algebraic structure defined in Section 3. The construction of the root groups utilizes the stabilizer theorem proved in Section 5. In Section 7 we construct the Tits geometries and count the points. Finally in Section 8 we derive the group orders, prove simplicity, and describe the exceptional behaviour of the first group in each series. We conclude the paper with some remarks on maximal subgroups, and on extending the constructions to infinite fields.

2. *Number systems and root systems*

The Gaussian integers are the elements of the ring $\mathcal{G} = \mathbb{Z}[i]$ of complex numbers, where $i^2 = -1$. The Eisenstein integers are the elements of $\mathcal{E} = \mathbb{Z}[\omega]$, where $\omega^2 + \omega + 1 = 0$. The Hurwitz ring of integral quaternions is $\mathcal{H} = \mathbb{Z}[i, \omega]$, where $\omega = \frac{1}{2}(-1 + i + j + k)$. As lattices, these are the root lattices of types B_2, G_2 and F_4 respectively. From these three rings we shall construct the Suzuki groups ${}^2B_2(2^{2n+1})$, the small Ree groups ${}^2G_2(3^{2n+1})$, and the large Ree groups ${}^2F_4(2^{2n+1})$.

The unit groups of these three rings are respectively

$$\begin{aligned} U(\mathcal{G}) &= \{\pm 1, \pm i\} \cong C_4, \\ U(\mathcal{E}) &= \{\pm 1, \pm\omega, \pm\overline{\omega}\} \cong C_6, \\ U(\mathcal{H}) &= \{\pm 1, \pm i, \pm j, \pm k, \tfrac{1}{2}(\pm 1 \pm i \pm j \pm k)\} \cong \mathrm{SL}_2(3), \end{aligned} \tag{2.1}$$

where $i^2 = j^2 = k^2 = -1$, $ij = -ji = k$, $i^\omega = j$, $j^\omega = k$ and $k^\omega = i$. To facilitate calculations in this last case it is useful to note the following identities:

$$\begin{aligned} \omega^i &= j\omega = \omega k = \tfrac{1}{2}(-1+i-j-k) \\ \omega^j &= k\omega = \omega i = \tfrac{1}{2}(-1-i+j-k) \\ \omega^k &= i\omega = \omega j = \tfrac{1}{2}(-1-i-j+k) \\ -\overline{\omega}^i &= \overline{\omega} j = k\overline{\omega} = \tfrac{1}{2}(1+i-j-k) \\ -\overline{\omega}^j &= \overline{\omega} k = i\overline{\omega} = \tfrac{1}{2}(1-i+j-k) \\ -\overline{\omega}^k &= \overline{\omega} i = j\overline{\omega} = \tfrac{1}{2}(1-i-j+k) \end{aligned} \tag{2.2}$$

In each case denote the set of units by U. Geometrically, these units form the short roots of a root system of type B_2, G_2, or F_4 respectively (i.e. a root system of type A_1A_1, A_2 or D_4, respectively). Then the set of long roots is the set of non-zero non-units of smallest norm, which is $(1+i)U$ in the cases $\mathcal{G}$ and $\mathcal{H}$, and is θU, where $\theta = \omega - \overline{\omega} = \sqrt{-3}$, in the case $\mathcal{E}$. Denote this set by L in each case.

We choose once and for all a linear map ϕ from U to L, which squares to a scalar p (where $p = 2$ in the cases $\mathcal{G}$ and $\mathcal{H}$, and $p = 3$ in the case $\mathcal{E}$), as follows.

$$\begin{aligned} &\phi : z \mapsto (1+i)\overline{z} \text{ in the case } \mathcal{G} \\ &\phi : z \mapsto (1-\overline{\omega})\overline{z} \text{ in the case } \mathcal{E} \\ &\phi : z \mapsto (1+i)z^j \text{ in the case } \mathcal{H} \end{aligned} \tag{2.3}$$

Since $\phi^2 = p$, the eigenvalues of ϕ are $\pm\sqrt{p}$. In the cases $\mathcal{G}$ and $\mathcal{E}$, both eigenspaces are 1-dimensional, while in the case $\mathcal{H}$ they are 2-dimensional. Explicit calculation shows that the short roots r are of two or three types, according to the inner product of r with $\phi(r)$.

Definition 1. *A root r is called*
(i) inner *if $r.\phi(r) = -p/2$,*
(ii) middle *if $r.\phi(r) = 0$, and*
(iii) outer *if $r.\phi(r) = p/2$.*

The reason for the terms inner, middle and outer will become clear once we have drawn pictures of the root systems. In the case $\mathcal{G}$ there are no middle roots, and we have
(1) if $r \in \{\pm 1\}$, then $r.\phi(r) = 1$, so r is outer; and
(2) if $r \in \{\pm i\}$, then $r.\phi(r) = -1$, so r is inner.
In the case $\mathcal{E}$ the three types are as follows.
(1) if $r \in \{\pm 1\}$, then $r.\phi(r) = 3/2$, so r is outer;
(2) if $r \in \{\pm\omega\}$, then $r.\phi(r) = -3/2$, so r is inner; and
(3) if $r \in \{\pm\overline{\omega}\}$, then $r.\phi(r) = 0$, so r is middle.
Finally in the case $\mathcal{H}$ we have
(1) if $r \in \{\pm 1, \pm j, \pm\overline{\omega}, \pm\overline{\omega}^i\}$, then $r.\phi(r) = 1$, so r is outer;
(2) if $r \in \{\pm i, \pm k, \pm\overline{\omega}^j, \pm\overline{\omega}^k\}$, then $r.\phi(r) = -1$, so r is inner; and
(3) if $r \in \{\pm\omega, \pm\omega^i, \pm\omega^j \pm \omega^k\}$, then $r.\phi(r) = 0$, so r is middle.
We next choose an ordering of the roots compatible with the map ϕ, ordering by the inner product with a suitable vector v_0.

Definition 2.
(i) *In the case B_2, let $v_0 = 2+i$; in the case G_2, let $v_0 = 4-\overline{\omega}$; and in the case F_4, let $v_0 = 8+3i+2j+k$.*
(ii) *If r is a root and $r.v_0 > 0$ we call the root* positive *and if $r.v_0 < 0$ the root is* negative.
(iii) *If $r.v_0 > s.v_0$ we write $r > s$.*

In the case B_2 the short roots are put in the order

$$-1, -i, i, 1. \tag{2.4}$$

In the case G_2 the ordering of short roots together with 0 is

$$-1, \overline{\omega}, \omega, 0, -\omega, -\overline{\omega}, 1. \tag{2.5}$$

Note that the ordering on long roots may also be obtained by applying ϕ to the ordering on short roots.

In the case F_4 we obtain only a partial order, and in cases where two short roots have the same inner product with v_0, we order them according to the order of the corresponding long roots. In cases where this does not discriminate, we make an arbitrary choice. Our ordering on the negative short roots is

$$-1, \overline{\omega}, \omega^k, \omega^j, \overline{\omega}^i, \omega^i, -i, \overline{\omega}^j, -j, \overline{\omega}^k, -k, \omega, \tag{2.6}$$

and on the positive short roots

$$-\omega, k, -\overline{\omega}^k, j, -\overline{\omega}^j, i, -\omega^i, -\overline{\omega}^i, -\omega^j, -\omega^k, -\overline{\omega}, 1. \tag{2.7}$$

Applying ϕ to these gives the ordering on long roots.

We end this section with some pictures. First we exhibit the cases $\mathcal{G}$ and $\mathcal{E}$ in full detail, in Fig. 1 and Fig. 2 respectively. Then we give the case $\mathcal{H}$ in its projection onto the $\sqrt{2}$-eigenspace of ϕ. This last is given in two versions. The first version, in Fig. 3, includes only the short roots, for clarity, while the second, in Fig. 4, includes also the long roots, for completeness. To construct these pictures of $\mathcal{H}$, we start by putting the eight inner short roots on the vertices of two superimposed squares. The relative position of these two squares can be determined by a small calculation. Then the positions of the middle roots are determined as they are sums of adjacent inner roots, and similarly the outer roots are the sums of three consecutive inner roots. The long roots are similarly the sums of pairs of perpendicular short roots. (It is perhaps worth remarking that if we project instead onto the $-\sqrt{2}$-eigenspace of ϕ, then we obtain a similar picture, but with the positions of the inner and outer roots interchanged.)

For convenience in performing these calculations, we list here the triples of short roots (r, s, t) with $r + s + t = 0$ (up to an overall sign).

$$\begin{array}{r}
1+\omega+\overline{\omega} = 1+\omega^i+\overline{\omega}^i = 1+\omega^j+\overline{\omega}^j = 1+\omega^k+\overline{\omega}^k = 0\\
i-\omega+\overline{\omega}^i = i-\omega^i+\overline{\omega} = i+\omega^j-\overline{\omega}^k = i+\omega^k-\overline{\omega}^j = 0\\
j-\omega+\overline{\omega}^j = j-\omega^j+\overline{\omega} = j+\omega^k-\overline{\omega}^i = j+\omega^i-\overline{\omega}^k = 0\\
k-\omega+\overline{\omega}^k = k-\omega^k+\overline{\omega} = k+\omega^i-\overline{\omega}^j = k+\omega^i-\overline{\omega}^k = 0
\end{array} \tag{2.8}$$

Notice that in each picture the ordering of the roots is from left to right, and from bottom to top. In the case $\mathcal{H}$, the inner, outer and middle roots lie on three regular octagons, which are respectively inner, outer and middle in the picture.

3. $\mathbb{W}$-*algebras*

We use the set U of units as an indexing set for a basis of a vector space, augmented by a set Z of 'zero' elements defined in the three cases by

$$\begin{aligned}
Z(\mathcal{G}) &= \{0\},\\
Z(\mathcal{E}) &= \{0, -0\},\\
Z(\mathcal{H}) &= \{0, \omega 0, \overline{\omega} 0\}.
\end{aligned} \tag{3.1}$$

Write $I = U \cup Z$. Let F be a field of characteristic p (where, as above, $p = 2$, 3, or 2 respectively). Let W be the vector space over F spanned by vectors e_t, for $t \in I$, subject to the relation $\sum_{t\in Z} e_t = 0$. Then W has dimension 4, 7 or 26 respectively. We shall specify the dimension by

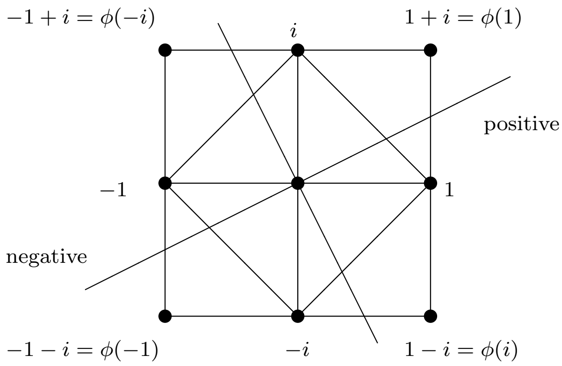


FIGURE 1. *The root system of type B_2*

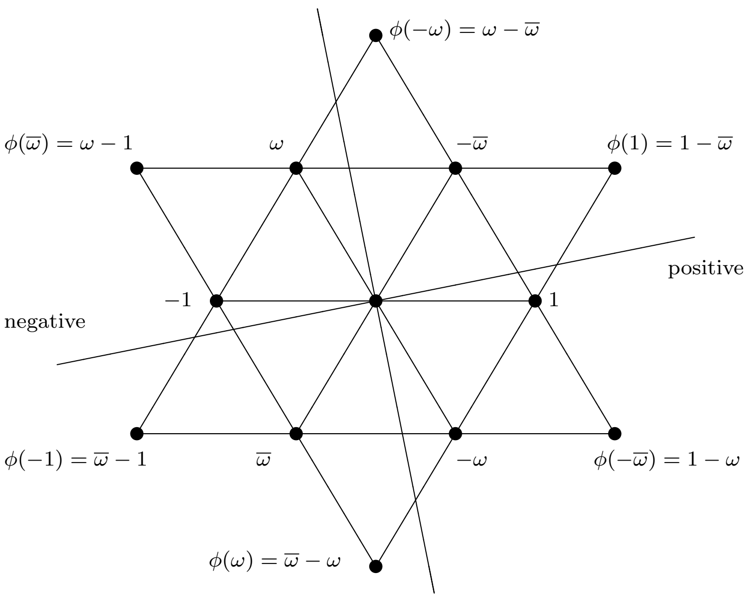


FIGURE 2. *The root system of type G_2*

writing W_4, W_7 or W_{26} for W when necessary. To prevent the notation becoming unreadable, we shall when necessary write $e(r)$ for e_r, and $E(r,s,\ldots)$ for $\langle e_r, e_s, \ldots\rangle$. We shall also write $E(S)$ for $\langle e(s) \mid s \in S\rangle$.

Using the ordering on the roots defined above, we may talk about the *leading term* of a vector in W (with a slight ambiguity, which will not be important, in the case of W_{26} if the leading term is one of the 'zero' terms $e_0, e_{\omega 0}, e_{\overline{\omega}0}$).

Roughly speaking, we shall put three products onto W, one an 'inner' or 'dot' product defined by pairs of short roots which sum to zero, the second an 'outer' or 'cross' product defined by pairs of short roots which sum to another short root, and the third a 'middle' or 'star' product defined by pairs of short roots which sum to a long root. The rest of Section 3 is devoted to making this precise.

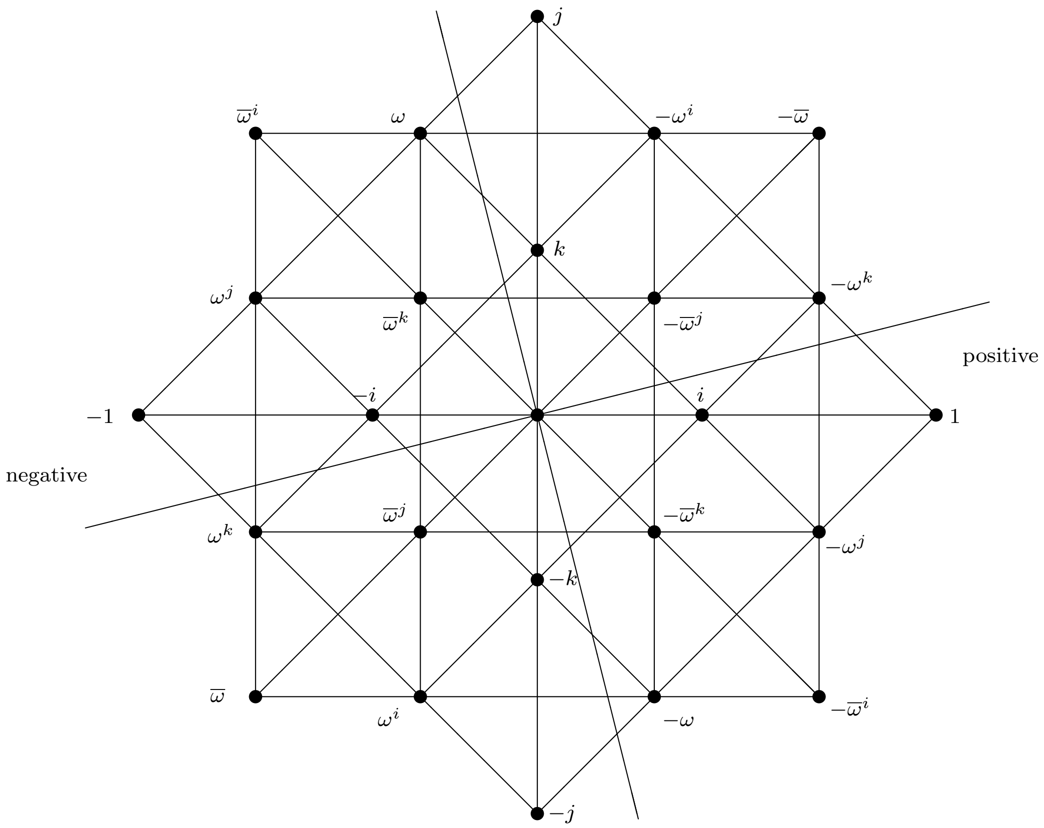


FIGURE 3. *The D_4 root system projected onto the $\sqrt{2}$-eigenspace of ϕ*

3.1. *The inner or dot product*

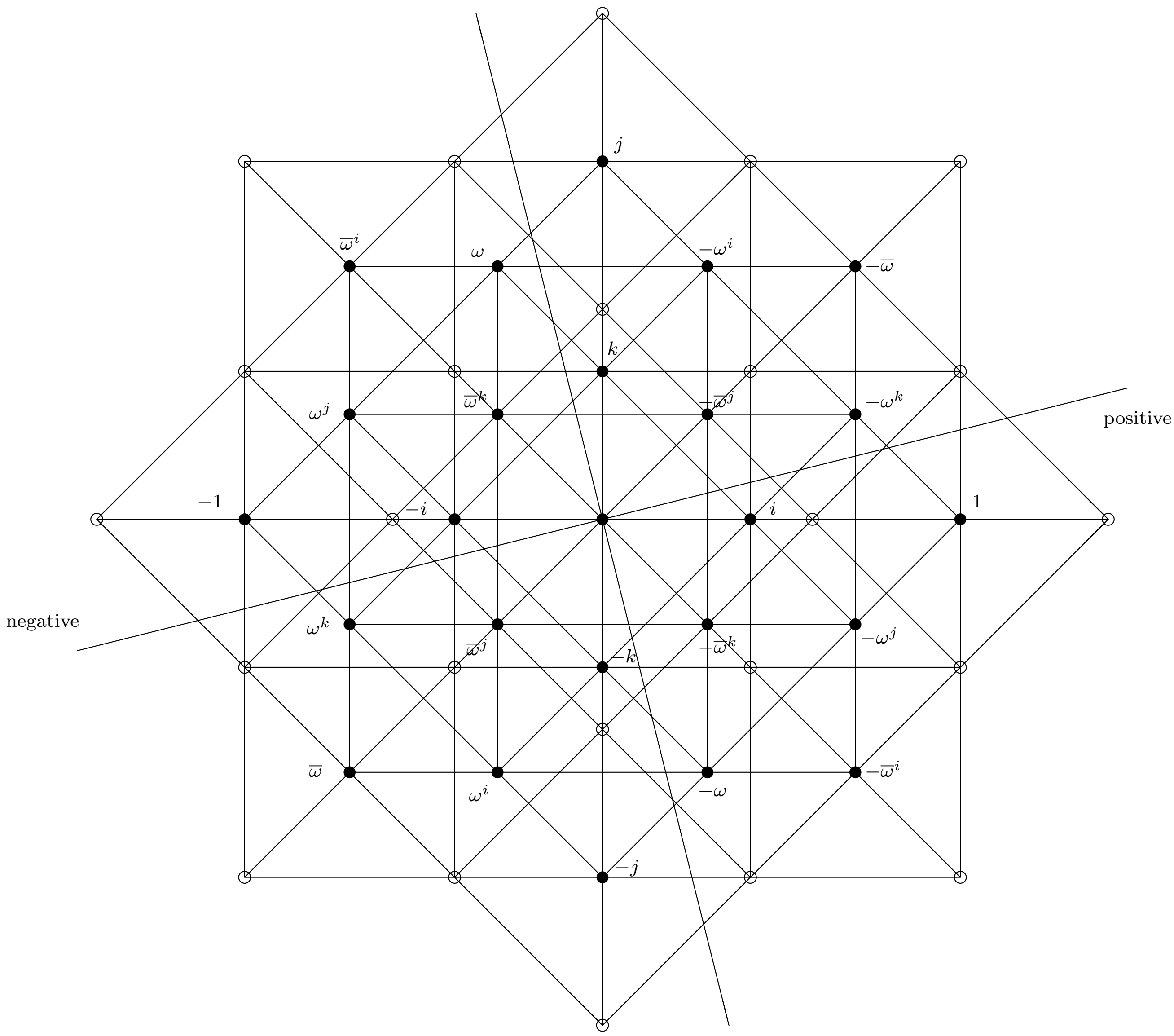
The inner product is a symmetric bilinear form $B : W \times W \to F$, where we also write $v.w$ for $B(v, w)$. It is defined by $B(e_t, e_{-t}) = 1$ for $t \in U$, and in the case W_7 also $B(e_0, e_0) = 1$, and in the case W_{26} also $B(e_t, e_{\omega t}) = 1$ for $t \in Z$, and in all cases $B(e_s, e_t) = 0$ otherwise. In the characteristic 2 cases, namely W_4 and W_{26}, the form B is also alternating, that is $B(v, v) = 0$. In the characteristic 3 case, namely W_7, the bilinear form B is equivalent to a quadratic form Q. On W_{26}, the form B is the bilinear form associated to a quadratic form Q, which may be defined by its values on a basis by $Q(e_t) = 0$ for $t \in U$ and $Q(e_t) = 1$ for $t \in Z$. It turns out that any linear map which preserves both the inner and outer products also preserves this quadratic form. However, this is not necessary for our theory, and so we shall not use the quadratic form in this case.

3.2. *The outer or cross product*

The outer product is an alternating (and therefore also skew-symmetric) bilinear product $M : W \times W \to W$. We shall write $v \times w$ for $M(v, w)$. This product has the following properties.

$$\begin{aligned} &E(r) \times E(s) = E(r + s) \text{ whenever } r, s, r + s \in U, \\ &E(r) \times E(s) = 0 \text{ if } r, s \in U, r + s \notin U. \end{aligned} \tag{3.2}$$

In the case W_4, there is no pair of short roots whose sum is a short root, so the outer product is identically zero. In the case W_7, all such sums derive from the equation $1 + \omega + \overline{\omega} = 0$ by taking one or two terms across to the right-hand side. As the characteristic is 3 there is a



The short roots are marked with black circles, and labelled with the corresponding Hurwitz integer. The long roots are marked with white circles, and unlabelled. The labels can be calculated (a) as $\phi(r)$ where r is a short root, using the fact that ϕ is multiplication by $\sqrt{2}$ in the picture, and (b) as $r + s$ where r and s are perpendicular short roots, by using the rectangular grid.

FIGURE 4. *The F_4 root system projected onto the $\sqrt{2}$-eigenspace of ϕ*

delicate question about the signs. We specify that $e_r \times e_s = e_t$ when r, s, t are in anti-clockwise order, and $e_r \times e_s = -e_t$ when the order is clockwise. When one of r, s, t is 0, we define the outer product by

$$\begin{aligned} e_{-1} \times e_1 &= e_0, \\ e_1 \times e_0 &= e_1, \\ e_0 \times e_{-1} &= e_{-1}, \end{aligned} \tag{3.3}$$

and images under multiplication of the subscripts by ω and $\overline{\omega}$. This product may be identified with the usual octonion product (modulo the centre) on the 7-space of pure imaginary octonions in characteristic 3. See for example Section 4.5.2 of [**21**].

In the case W_{26}, the outer product is equivalent to the product on the trace 0 part of the exceptional Jordan algebra. The products which do not involve any zero subscripts are of the form $e_r \times e_s = e_{r+s}$, which may be more symmetrically written $e_r \times e_s = e_{-t}$ whenever $r, s, t \in U$ satisfy $r + s + t = 0$. The triples which occur have already been listed in (2.8) and can also be read off from Fig. 3.

In the case when one of r, s, t is zero we have to distinguish carefully between the three different zeroes, 0, $\omega 0$ and $\overline{\omega} 0$. The short roots fall into three cosets Q_8, ωQ_8 and $\overline{\omega} Q_8$ of the quaternion group $Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$. We adopt the convention that for r in one of these three cosets, $r + (-r) = 0$ or $\omega 0$ or $\overline{\omega} 0$ respectively. The rest of the values of the outer product are now given by

$$\begin{aligned} e_0 \times e_{\omega 0} &= 0, \\ e_r \times e_{-r} &= e_0, e_{\omega 0}, e_{\overline{\omega} 0} \text{ according as } r \in Q_8, \omega Q_8, \overline{\omega} Q_8, \\ e_0 \times e_r &= e_r \text{ when } r \in \omega Q_8 \cup \overline{\omega} Q_8, \\ e_{\omega 0} \times e_r &= e_r \text{ when } r \in Q_8 \cup \overline{\omega} Q_8, \\ e_{\overline{\omega} 0} \times e_r &= e_r \text{ when } r \in Q_8 \cup \omega Q_8. \end{aligned} \tag{3.4}$$

3.3. *The trilinear form*

The inner and outer products together give rise to a skew-symmetric trilinear form T defined by $T(u, v, w) = (u \times v).w$. It is easy to check that this is cyclically symmetric on the basis vectors. Indeed, the non-zero values at basis vectors occur for $T(e_r, e_s, e_t)$ where $r + s + t = 0$. In the case W_4, of course, T is the zero form, as the outer product is zero.

In the case W_7, either r, s, t are all non-zero, and we have $T(e_r, e_{\omega r}, e_{\overline{\omega} r}) = 1$, or one of them is zero, and we have $T(e_r, e_{-r}, e_0) = 1$ for $r = 1, \omega, \overline{\omega}$. (For these values of r, we adopt the convention that $r + (-r) = 0$ while $(-r) + r = -0$.)

In the case W_{26}, where the characteristic is 2, we have

$$\begin{aligned} T(e_r, e_s, e_t) &= 1 \text{ whenever } r, s, t \text{ are non-zero and } r + s + t = 0. \\ T(e_r, e_{-r}, e_0) &= 1 \text{ for } r \in \omega Q_8 \cup \overline{\omega} Q_8, \\ T(e_r, e_{-r}, e_{\omega 0}) &= 1 \text{ for } r \in Q_8 \cup \overline{\omega} Q_8, \\ T(e_r, e_{-r}, e_{\overline{\omega} 0}) &= 1 \text{ for } r \in Q_8 \cup \omega Q_8, \\ T(e_0, e_{\omega 0}, e_{\overline{\omega} 0}) &= 1, \end{aligned} \tag{3.5}$$

and $T(e_r, e_s, e_t) = 0$ otherwise.

3.4. *The middle or star product*

When r and s are two short roots whose sum is a long root, we have that

$$t = \phi^{-1}(r + s) = \phi(r + s)/p \tag{3.6}$$

is a short root, and we define $e_r \star e_s = e_t$ (with the condition $r = 1, \omega, \overline{\omega}$ in the case W_7). We also define

$$e_r \star e_{-r} + e_s \star e_{-s} = e_{t+(-t)}, \tag{3.7}$$

with the same conventions as above for the different types of zeroes. For all other pairs of basis vectors we define $e_r \star e_s = 0$.

Now we extend this product by the rules

$$\begin{aligned} u \star v &= -v \star u \\ u \star (v + w) &= u \star v + u \star w \\ u \star (\lambda v) &= \lambda^\sigma (u \star v) \end{aligned} \tag{3.8}$$

where $\sigma^{-1} = \tau$ is an automorphism of F which squares to the Frobenius automorphism $\lambda \mapsto \lambda^p$. This last condition implies that the field F must have order p^{2n+1}, and then $\lambda^\sigma = \lambda^{p^n}$ and $\lambda^\tau = \lambda^{p^{n+1}}$.

For the purposes of defining the groups, however, we must restrict this product to pairs of isotropic vectors u, v which satisfy $u.v = 0$ and $u \times v = 0$. Observe that since $e_r \star e_r = 0$, the anti-symmetry implies that $v \star v = 0$ for all isotropic v. A more formal way to define this product, which perhaps makes it clearer that it is really well-defined, is to first interpret the dot and cross products as linear maps $\pi_1 : W \wedge W \to F$ and $\pi_2 : W \wedge W \to W$, and then to define $\pi_3 : (\ker \pi_1) \cap (\ker \pi_2) \to W$ by interpreting $u \star v$ as $\pi_3(u \wedge v)$ and $u \star v + w \star x$ as $\pi_3(u \wedge v + w \wedge x)$.

It may be useful to list here the non-trivial star products in each case. In W_4 we have

$$\begin{aligned} e_1 \star e_i &= e_1 \\ e_1 \star e_{-i} &= e_i \end{aligned} \tag{3.9}$$

and images under negating the subscripts. (Notice incidentally that if $e_r \star e_s = e_t$, then $e_{ir} \star e_{is} = e_{-it}$.) In W_7 we have

$$\begin{aligned} e_1 \star e_{-\overline{\omega}} &= e_1 \\ e_\omega \star e_{-1} &= e_{\overline{\omega}} \\ e_{\overline{\omega}} \star e_{-\omega} &= e_\omega \\ e_1 \star e_{-1} + e_{-\omega} \star e_\omega &= e_0 \\ e_{\overline{\omega}} \star e_{-\overline{\omega}} + e_{-1} \star e_1 &= e_0 \end{aligned} \tag{3.10}$$

In this case when we negate the subscripts we also negate e_0, since $e_{-0} = -e_0$. We also have that if $e_r \star e_s = e_t$ then $e_{\omega r} \star e_{\omega s} = e_{\overline{\omega} t}$.

In the case of W_{26} we may use Fig. 4 to read off the products. We take two short roots r, s, corresponding to black circles in the figure, with the property that their sum is a long root, corresponding to a white circle. This white circle is found by usual vector addition. Then we shrink the result by a factor of $\sqrt{2}$ until it becomes a short root t, say: we now have $e_r \star e_s = e_t$. For example, if $r = 1$ and $s = k$ then $r + s$ shrinks down to $t = -\omega^k$ and we have $e_1 \star e_k = e_{-\omega^k}$. We give these products here in a simplified notation, so that an entry t in row r and column s denotes that $e_r \star e_s = e_t$. The products which are not explicitly listed can be read off from the fact that if $e_r \star e_s = e_t$ then $e_{-r} \star e_{-s} = e_{-t}$. First we give the product of roots in Q_8.

$r\backslash s$	-1	$-i$	$-j$	$-k$	k	j	i	1
-1		-1	$\overline{\omega}$	ω^k	ω^j	$\overline{\omega}^i$	$-i$	
$-i$	-1		ω^i	$\overline{\omega}^j$	$\overline{\omega}^k$	ω		i
$-j$	$\overline{\omega}$	ω^i		$-j$	$-k$		$-\omega$	$-\overline{\omega}^i$
$-k$	ω^k	$\overline{\omega}^j$	$-j$			k	$-\overline{\omega}^k$	$-\omega^j$

(3.11)

Next we give the product of roots in ωQ_8.

$$\begin{array}{c|cccc|cccc|}
r\backslash s & \omega^k & \omega^j & \omega^i & \omega & -\omega & -\omega^i & -\omega^j & -\omega^k \\ \hline
\omega^k & & -1 & \overline{\omega} & \omega^j & \omega^i & \overline{\omega}^k & -k & \\
\omega^j & -1 & & \omega^k & \overline{\omega}^i & \overline{\omega}^j & \omega & & k \\
\omega^i & \overline{\omega} & \omega^k & & -i & -j & & -\omega & -\overline{\omega}^k \\
\omega & \omega^j & \overline{\omega}^i & -i & & & j & -\overline{\omega}^j & -\omega^i \\ \hline
\end{array} \tag{3.12}$$

Finally we give the product of roots in $\overline{\omega}Q_8$.

$$\begin{array}{c|cccc|cccc|}
r\backslash s & \overline{\omega} & \overline{\omega}^i & \overline{\omega}^j & \overline{\omega}^k & -\overline{\omega}^k & -\overline{\omega}^j & -\overline{\omega}^i & \overline{\omega} \\ \hline
\overline{\omega} & & -1 & \overline{\omega} & \omega^k & \omega^i & \overline{\omega}^j & -j & \\
\overline{\omega}^i & -1 & & \omega^j & \overline{\omega}^i & \overline{\omega}^k & \omega & & j \\
\overline{\omega}^j & \overline{\omega} & \omega^j & & -i & -k & & -\omega & -\overline{\omega}^j \\
\overline{\omega}^k & \omega^k & \overline{\omega}^i & -i & & & k & -\overline{\omega}^k & -\omega^i \\ \hline
\end{array} \tag{3.13}$$

In addition, the products involving the zero elements are

$$\begin{array}{l}
e_r \star e_{-r} + e_s \star e_{-s} = e_0 \text{ when } r, s \in Q_8 \\
e_r \star e_{-r} + e_s \star e_{-s} = e_{\omega 0} \text{ when } r, s \in \omega Q_8 \\
e_r \star e_{-r} + e_s \star e_{-s} = e_{\overline{\omega} 0} \text{ when } r, s \in \overline{\omega} Q_8
\end{array} \tag{3.14}$$

3.5. *Definitions of the automorphism groups*

In each case let us define the 'algebra' $\mathbb{W}$ to be the vector space W endowed with the three products just defined. As before, we add a subscript to indicate the dimension when necessary.

DEFINITION 3. *An* automorphism *of* $\mathbb{W}$ *is a linear map g which preserves the three products, in the sense that*

(i) $u^g.v^g = u.v$ *for all* $u, v \in W$;
(ii) $u^g \times v^g = (u \times v)^g$ *for all* $u, v \in W$; *and*
(iii) $u^g \star v^g = (u \star v)^g$ *for all* $u, v \in W$ *which satisfy* $u.u = u.v = v.v = 0$ *and* $u \times v = 0$.

With this definition of the automorphism groups, it turns out that the automorphism groups of $\mathbb{W}_4$ are isomorphic to the Suzuki groups, the automorphism groups of $\mathbb{W}_7$ are isomorphic to the small Ree groups, and the automorphism groups of $\mathbb{W}_{26}$ are isomorphic to the large Ree groups. Indeed, one could reasonably take these are the definitions of the Suzuki and Ree groups.

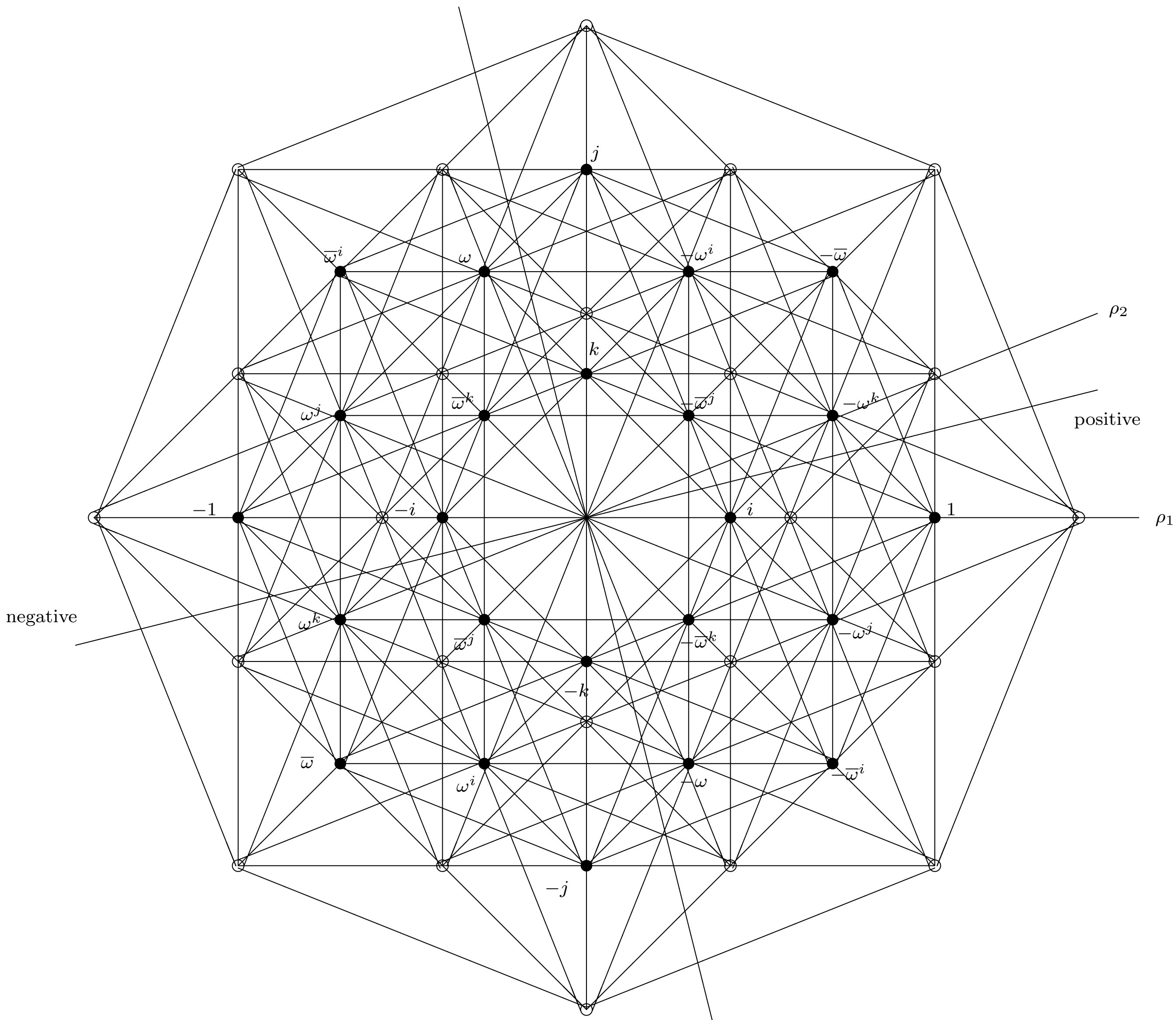
4. *The Weyl group and the torus*

In this section we shall exhibit some elements of the automorphism groups of the $\mathbb{W}$-algebras, which will eventually turn out to be sufficient to generate them. In increasing order of difficulty these are generators for the Weyl group (that is, the group of coordinate permutations), the maximal torus (that is, the group of diagonal matrices), and the root groups (that is, certain groups of lower triangular matrices).

4.1. *The Weyl group*

The Weyl group of our root system, of type B_2, G_2 or F_4, is by definition the group generated by the reflections in the roots. If r is a short root, so that $r\overline{r} = 1$, then reflection in r is the map $z \mapsto -r\overline{z}r$, while if r is a long root, so that $r\overline{r} = p$, it is the map $z \mapsto -r\overline{z}r/p$.

DEFINITION 4. *The* twisted Weyl group *is the subgroup of the Weyl group which commutes with* ϕ.



The axes of the fundamental reflections ρ_1 and ρ_2 of the Weyl group are marked on the picture, as is a line separating positive and negative roots, and the direction along which the ordering of the roots is measured. The outer roots mark (the leading terms of) the points, and the edges making these into an octagon mark the lines, of the generalized octagon.

FIGURE 5. *The F_4 root system showing the octagonal symmetry*

It is easy to see that in the case of B_2 the Weyl group is the dihedral group D_8 of order 8, and in the case of G_2 it is $D_{12} \cong 2 \times S_3$. In these two cases it is obvious that the part of the Weyl group which commutes with ϕ is just the group of order 2 generated by $t \mapsto -t$. This now acts on $\mathbb{W}$ by $e_t \mapsto e_{-t}$ (including $e_0 \mapsto e_{-0} = -e_0$ in the case $\mathbb{W}_7$), and clearly preserves the forms B and T, as well as the partial product $\star$. In both cases this group C_2 is transitive on roots r with a given value of $r.\phi(r)$.

In the case F_4 the full (untwisted) Weyl group is a group of order $2^7.3^2 = 1152$ and shape $2^{1+4}.(S_3 \times S_3)$, and the subgroup which commutes with ϕ is a dihedral group D_{16}, although we shall not need any of these facts. All we need is that ϕ commutes with the dihedral group D_{16} generated by the maps

$$\begin{aligned}\rho_1 &: z \mapsto z^i\\ \rho_2 &: z \mapsto (1+i)z^k(1+j)/2.\end{aligned} \tag{4.1}$$

These maps induce linear maps on $\mathbb{W}_{26}$ via $\rho : e_r \mapsto e_{\rho(r)}$, and by defining ρ_1 to fix the vectors e_t with $t \in Z$, while ρ_2 swaps e_0 with $e_{\overline{\omega}0}$. More explicitly, ρ_2 acts by permuting the coordinates as

$$\begin{aligned}\rho_2 = {} & (0, \overline{\omega}0)(1, -\overline{\omega})(i, -\overline{\omega}^j)(j, -\overline{\omega}^i)(k, -\overline{\omega}^k)\\ & (-1, \overline{\omega})(-i, \overline{\omega}^j)(-j, \overline{\omega}^i)(-k, \overline{\omega}^k)\\ & (\omega^i, \omega^j)(-\omega^i, -\omega^j)(\omega, -\omega).\end{aligned} \tag{4.2}$$

Again, it is easy to see that these maps preserve all the forms and products. And again, the twisted Weyl group is transitive on roots r with a fixed value of $r.\phi(r)$. In Fig. 5 we show the root system with the full octagonal symmetry under the action of the Weyl group.

4.2. *The maximal torus*

Consider a diagonal symmetry $e_t \mapsto \lambda_t e_t$. Since this preserves the bilinear form B it must satisfy $\lambda_t\lambda_{-t} = 1$, which implies that $\lambda_{-t} = {\lambda_t}^{-1}$ for all short roots t. Since it preserves the trilinear form T, which has non-zero terms $T(e_r, e_{-r}, e_{r+(-r)})$ we also get $\lambda_t = 1$ for $t \in Z$. Also $T(e_r, e_s, e_t)$ is non-zero whenever r, s, t are short roots with $r + s + t = 0$, so we obtain corresponding equations $\lambda_r\lambda_s\lambda_t = 1$. Since the equations $r+s+t = 0$ are sufficient to define the ambient 2- or 4-dimensional space in which the root system lies, the corresponding equations are sufficient to reduce the number of free parameters λ_r to 2 (in the cases $\mathbb{W}_4$ and $\mathbb{W}_7$) or 4 (in the case $\mathbb{W}_{26}$). For example we may take free parameters λ_r as r runs over a system of fundamental roots for the system of short roots, say $\{1, i\}$ for the system of type A_1A_1 in the first case, or $\{1, \omega\}$ for the system of type A_2 in the second, or $\{1, i, j, \overline{\omega}\}$ for the system of type D_4 in the third.

Finally, to preserve the product $\star$ where $e_r \star e_s = e_{\phi^{-1}(r+s)}$ it must satisfy the condition

$$(\lambda_r\lambda_s)^\sigma = \lambda_{\phi^{-1}(r+s)}, \tag{4.3}$$

which can also be written

$$\lambda_r\lambda_s = (\lambda_{\phi^{-1}(r+s)})^\tau. \tag{4.4}$$

We shall show that this gives one condition on the two free parameters in the first two cases, and two conditions on the four free parameters in the last case.

To see that the many different equations given here are consistent, we need to use the fact that $\tau^2 = p$, in the sense that τ^2 is the Frobenius automorphism. Explicitly, in the case $\mathbb{W}_4$ we have $\lambda_1\lambda_i = {\lambda_1}^\tau$, which we can write as $\lambda_i = {\lambda_1}^{\tau-1}$, which is equivalent to $\lambda_1 = (\lambda_i)^{\tau+1}$ since $(\tau-1)(\tau+1) = \tau^2 - 1 = 1$. Thus it is equivalent to the other equation $\lambda_1\lambda_{-i} = (\lambda_i)^\tau$.

Similarly, in the case $\mathbb{W}_7$, the three equations are

$$\begin{aligned}\lambda_1\lambda_{-\overline{\omega}} &= {\lambda_1}^\tau\\ \lambda_\omega\lambda_{-1} &= {\lambda_{\overline{\omega}}}^\tau\\ \lambda_{\overline{\omega}}\lambda_{-\omega} &= {\lambda_\omega}^\tau\end{aligned} \tag{4.5}$$

and since $\lambda_1\lambda_\omega\lambda_{\overline{\omega}} = 1$, any two of these equations imply the third. Moreover, substituting $\lambda_{\overline{\omega}} = \lambda_{-1}\lambda_{-\omega}$ into the first equation gives $\lambda_\omega = {\lambda_1}^{-2+\tau}$, and substituing into the third equation gives $\lambda_{-1} = {\lambda_\omega}^{\tau+2}$, which is equivalent since $(\tau+2)(\tau-2) = \tau^2 - 4 = -1$. Hence these three equations are equivalent.

Finally in the case $\mathbb{W}_{26}$ we have

$$\begin{aligned} \lambda_1\lambda_i &= {\lambda_1}^{\tau} \\ \lambda_1\lambda_j &= {\lambda_{\overline{\omega}}}^{-\tau} \end{aligned} \tag{4.6}$$

so we can take the two free parameters to be λ_1 and $\lambda_{\overline{\omega}}$, and express all the other parameters in terms of them. Now every root can be expressed as

$$a.1 + b.\phi(1) + c.\overline{\omega} + d.\phi(\overline{\omega}), \tag{4.7}$$

where a, b, c, d are integers, and the corresponding eigenvalue is

$$(\lambda_1)^{a+b\tau}(\lambda_{\overline{\omega}})^{c+d\tau}. \tag{4.8}$$

In Fig. 5 this root is drawn at position $(a + b\sqrt{2}) + (c + d\sqrt{2})\overline{\omega}$. Moreover, adding exponents corresponds to adding vectors in Fig. 5, and multiplying exponents by τ corresponds to multiplying vectors by $\sqrt{2}$. Hence the geometry of the figure makes clear that the eigenvalues are well-defined by this procedure.

In conclusion, we have

Theorem 1.
(i) *The group of diagonal matrices which are automorphisms of $\mathbb{W}_4$ or $\mathbb{W}_7$ is a cyclic group of order $q - 1$;*
(ii) *The group of diagonal matrices which are automorphisms of $\mathbb{W}_{26}$ is $C_{q-1} \times C_{q-1}$.*

5. *A stabilizer theorem*

In order to motivate the construction of the root groups, we show that certain subgroups of the stabilizer of $E(-1)$ are diagonal. The proofs of these results actually give an effective algorithm to compute each root group explicitly. The main purpose of these results is however to help us prove that the automorphism group of $\mathbb{W}$ is generated by the root groups, together with the torus and the Weyl group. This is a crucial ingredient in the later calculation of the order of the automorphism group. In each case we fix $E(-1)$ and either the zero terms, or in the case $\mathbb{W}_4$ where there is no zero term, the one immediately above where the zero would be, and show that the only remaining automorphisms are diagonal.

Theorem 2. *Any automorphism of $\mathbb{W}_4$ which fixes $E(-1)$ and $E(i)$ lies in the diagonal subgroup, which is cyclic of order $q - 1$.*

Proof. Any such automorphism must fix both

$$\begin{aligned} E(-1) \star E(i) &= E(-i) \text{ and} \\ (E(i) \star W) \cap E(i)^{\perp} &= E(1), \end{aligned} \tag{5.1}$$

so is diagonal. We have just shown in Theorem 1 that the group of diagonal automorphisms is isomorphic to the multiplicative group of the field, so is cyclic of order $q - 1$. □

Theorem 3. *Any automorphism of $\mathbb{W}_7$ which fixes $E(-1)$ and $E(0)$ lies in the diagonal subgroup, which is cyclic of order $q - 1$.*

Proof. First note that the map $x \mapsto e_0 \times x$ has eigenvalues $-1, 0, 1$ with multiplicities $3, 1, 3$ respectively, and eigenspaces

$$\begin{aligned} W_- &= E(-1, -\omega, -\overline{\omega}) \\ W_0 &= E(0) \end{aligned}$$

$$W_+ = E(1, \omega, \overline{\omega}) \tag{5.2}$$

Now we have

$$E(-1) \star W = E(-1, \overline{\omega}), \tag{5.3}$$

whose intersection with W_+ determines $E(\overline{\omega})$. Then $E(\overline{\omega}) \star W = E(-1, \omega)$, whose intersection with W_+ determines $E(\omega)$. Next, $E(\omega) \star W = E(\overline{\omega}, -\omega)$, whose intersection with W_- determines $E(-\omega)$; then we have $E(-\omega) \star W = E(-\overline{\omega}, \omega)$ whose intersection with W_- determines $E(-\overline{\omega})$; and finally, $E(-\overline{\omega}) \star W = E(1, -\omega)$, whose intersection with W_+ determines $E(1)$. Therefore all the coordinate 1-spaces are determined, which means that the given automorphism is diagonal. In this case also we have shown that the group of diagonal automorphisms is cyclic of order $q-1$. □

THEOREM 4. *The subgroup of the automorphism group of $\mathbb{W}_{26}$ which fixes $E(-1)$, $E(0)$ and $E(\overline{\omega}0)$ has order $2(q-1)^2$ and is generated by the diagonal elements and ρ_1.*

Proof. Suppose that g is an automorphism of $\mathbb{W}_{26}$ which fixes $E(-1)$, $E(0)$ and $E(\overline{\omega}0)$. The space $E(0, \overline{\omega}0)^\perp$ is fixed, and is the space $E(U)$ spanned by all $e(r)$ for $r \in U$. On this 24-space, the map $v \mapsto v \times e(0)$ has kernel

$$E(Q_8) = E(\pm 1, \pm i, \pm j, \pm k), \tag{5.4}$$

which is therefore fixed. Similarly the kernels of the maps $v \mapsto v \times e(\omega 0)$ and $v \mapsto v \times e(\overline{\omega}0)$ are respectively

$$\begin{aligned} E(\omega Q_8) &= E(\pm\omega, \pm\omega^i, \pm\omega^j, \pm\omega^k) \\ E(\overline{\omega} Q_8) &= E(\pm\overline{\omega}, \pm\overline{\omega}^i, \pm\overline{\omega}^j, \pm\overline{\omega}^k), \end{aligned} \tag{5.5}$$

so both are fixed. For the rest of the proof it will be useful to refer to Fig. 4 (or Fig. 5) for the calculation of the various spaces $E(r) \star W$.

(i) First, the space

$$E(-1) \star W = E(-1, -i, \omega^j, \omega^k, \overline{\omega}, \overline{\omega}^i) \tag{5.6}$$

is fixed, and therefore so are the respective intersections $E(-1, -i)$, $E(\omega^j, \omega^k)$ and $E(\overline{\omega}, \overline{\omega}^i)$ with the spaces $E(Q_8)$, $E(\omega Q_8)$ and $E(\overline{\omega} Q_8)$. Now it is easy to see from the operation table for $\star$ that if $v \in E(\overline{\omega}, \overline{\omega}^i)$ satisfies $v = v \star w$ for some w then either $v \in E(\overline{\omega})$ or $v \in E(\overline{\omega}^i)$. But ρ_1 swaps $E(\overline{\omega})$ with $E(\overline{\omega}^i)$ while fixing $E(-1)$, $E(0)$ and $E(\overline{\omega}0)$, so we may assume that g fixes $E(\overline{\omega})$ and $E(\overline{\omega}^i)$.

(ii) Next calculate

$$\begin{aligned} E(\overline{\omega}) \star W &= E(-1, -j, \omega^k, \omega^i, \overline{\omega}, \overline{\omega}^j) \\ E(\overline{\omega}^i) \star W &= E(-1, j, \omega, \omega^j, \overline{\omega}^i, \overline{\omega}^k), \end{aligned} \tag{5.7}$$

and intersect with $E(\omega^j, \omega^k)$ to see that $E(\omega^k)$ and $E(\omega^j)$ are fixed.

(iii) Now calculate

$$\begin{aligned} E(\omega^k) \star W &= E(-1, -k, \omega^i, \omega^j, \overline{\omega}, \overline{\omega}^k) \\ E(\omega^j) \star W &= E(-1, k, \omega, \omega^k, \overline{\omega}^i, \overline{\omega}^j) \end{aligned} \tag{5.8}$$

and intersect with the fixed spaces already calculated to see that g fixes $E(\overline{\omega}^k)$ and $E(\overline{\omega}^j)$, and $E(\omega)$ and $E(\omega^i)$. It then follows that $E(-i) = E(\overline{\omega}^j) \star E(\overline{\omega}^k)$ is also fixed.

(iv) Now we can calculate

$$\begin{aligned} E(\overline{\omega}^k) \star W &= E(-i, k, \omega^k, -\omega^i, \overline{\omega}^i, -\overline{\omega}^k) \\ E(\overline{\omega}^j) \star W &= E(-i, -k, -\omega, \omega^j, \overline{\omega}, -\overline{\omega}^j) \\ E(\omega^i) \star W &= E(-i, -j, -\omega, \omega^k, \overline{\omega}, -\overline{\omega}^k) \end{aligned}$$

$$
\begin{aligned}
E(\omega) \star W &= E(-i, j, -\omega^i, \omega^j, \overline{\omega}^i, -\overline{\omega}^j) \\
E(-i) \star W &= E(-1, i, \omega, \omega^i, \overline{\omega}^j, \overline{\omega}^k)
\end{aligned}
\tag{5.9}
$$

and the various intersections give the fixed 1-spaces $E(k)$, $E(-j)$, $E(-\omega)$, $E(-\omega^i)$, $E(-\overline{\omega}^j)$ and $E(-\overline{\omega}^k)$.

(v) All the remaining coordinates can be calculated with the outer and star products, as follows:

$$
\begin{aligned}
E(j) &= E(\overline{\omega}^k) \times E(-\omega^i) \\
E(-k) &= E(\omega^i) \times E(-\overline{\omega}^j) \\
E(i) &= E(-\overline{\omega}^j) \star E(-\overline{\omega}^k) \\
E(-\overline{\omega}) &= E(i) \times E(-\omega^i) \\
E(-\omega^k) &= E(i) \times E(-\overline{\omega}^j) \\
E(-\omega^j) &= E(i) \times E(-\overline{\omega}^k) \\
E(-\overline{\omega}^i) &= E(i) \times E(-\omega) \\
E(1) &= E(-\omega^j) \times E(-\overline{\omega}^j)
\end{aligned}
\tag{5.10}
$$

Hence g is diagonal. We have already shown that the subgroup of diagonal elements is the torus $D \cong C_{q-1} \times C_{q-1}$, so this concludes the proof. □

6. *Root elements*

The simplest non-monomial symmetries are the so-called 'root elements'. There is one type of root element for each orbit of the Weyl group on the roots.

6.1. *Root elements on $\mathbb{W}_4$*

In the case of $\mathbb{W}_4$, there are two types of roots, so two types of root elements. In fact, the root elements corresponding to the roots $\pm i$ square to root elements corresponding to roots ± 1, and the corresponding 'root subgroups' are special groups of order q^2.

In order to construct such a root subgroup, we shall prove that for any $\alpha, \beta \in F$ there is a unique symmetry $f_{\alpha,\beta}$ which fixes e_{-1} and maps $e_i \mapsto e_i + \alpha e_{-i} + \beta e_{-1}$. Uniqueness is immediate from the stabilizer theorem (Theorem 2) in the previous section.

To prove existence, it is sufficient to consider the case $\alpha = 1$, $\beta = 0$, since the element $f_{1,0}$ together with its conjugates by the maximal torus will then generate the whole root subgroup. The proof of Theorem 2 actually gives us an algorithm for constructing this element. Write e'_t for the image of e_t under $f_{1,0}$. Thus $e'_i = e_i + e_{-i}$, and therefore $e'_{-i} = e'_{-1} \star e'_i = e_{-i} + e_{-1}$. Then

$$
\begin{aligned}
e'_i \star W &= (e_i + e_{-i}) \star \langle e_{-1}, e_1 \rangle \\
&= \langle e_1 + e_i, e_{-i} + e_{-1} \rangle,
\end{aligned}
\tag{6.1}
$$

and using $e'_1.e'_{-i} = 0$ we have $e'_1 = e_1 + e_i + e_{-i} + e_{-1}$. In other words $f_{1,0}$ is represented with respect to the ordered basis $\{e_{-1}, e_{-i}, e_i, e_1\}$ by the matrix

$$
\begin{pmatrix}
1 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 \\
0 & 1 & 1 & 0 \\
1 & 1 & 1 & 1
\end{pmatrix}.
$$

It is a triviality to check that this element preserves the inner product. We have already checked the product $e'_{-1} \star e'_i$, and the case $e'_{-1} \star e'_{-i}$ is trivial, which leaves the three cases:

$$
\begin{aligned}
e'_{-i} \star e'_1 &= (e_{-i} + e_{-1}) \star (e_1 + e_i + e_{-i} + e_{-1}) \\
&= e_i + e_{-1} + e_{-i} + e_{-1} = e'_i \\
e'_1 \star e'_i &= (e_1 + e_i + e_{-i} + e_{-1}) \star (e_i + e_{-i})
\end{aligned}
$$

$$\begin{aligned} &= e_1 + e_i + e_{-i} + e_{-1} = e'_1 \\ e'_1 \star e'_{-1} + e'_i \star e'_{-i} &= (e_1 + e_i + e_{-i} + e_{-1}) \star e_{-1} + (e_i + e_{-i}) \star (e_{-i} + e_{-1}) \\ &= e_{-i} + e_{-1} + e_{-i} + e_{-1} = 0 \end{aligned} \tag{6.2}$$

as required. Notice that in this last case the individual terms $e'_1 \star e'_{-1}$ and $e'_i \star e'_{-i}$ are not zero, which is why we had to restrict the star product to pairs of perpendicular vectors.

6.2. *Root elements on* $\mathbb{W}_7$

In the case $\mathbb{W}_7$, there are three types of roots and therefore three types of root elements. In fact, the root elements corresponding to -1, $\overline{\omega}$ and ω together generate a root subgroup of order q^3. In all cases except $q = 3$, it is sufficient to construct the root element corresponding to ω.

Indeed, a similar calculation to the case $\mathbb{W}_4$ shows that for each $\alpha, \beta, \gamma \in F$ there is a unique symmetry $f_{\alpha,\beta,\gamma}$ which fixes e_{-1} and maps

$$e_0 \mapsto e_0 + \alpha e_\omega + \beta e_{\overline{\omega}} + \gamma e_{-1}.$$

Uniqueness is again immediate from the stabilizer theorem (Theorem 3) above.

To prove existence we apply the algorithm suggested by the proof of Theorem 3 to the case $\alpha = 1$, $\beta = \gamma = 0$. Write e'_t for the image of e_t under this map, so that $e'_{-1} = e_{-1}$ and $e'_0 = e_0 + e_\omega$. We first find the eigenspaces of the map $x \mapsto (e_0 + e_\omega) \times x$ to be

$$\begin{aligned} W'_- &= \langle e_{-1}, e_{-\overline{\omega}}, e_{-\omega} - e_0 + e_\omega \rangle \\ W'_0 &= \langle e_0 + e_\omega \rangle \\ W'_+ &= \langle e_\omega, e_1 - e_{-\overline{\omega}}, e_{-1} + e_{\overline{\omega}} \rangle \end{aligned} \tag{6.3}$$

Therefore $e'_{\overline{\omega}} = e_{\overline{\omega}} + e_{-1}$, since it lies in $e_{-1} \star W = \langle e_{-1}, e_{\overline{\omega}} \rangle$ and in W'_+. Now we calculate $v \star W$ for each v in turn: in each case we first calculate the 3-dimensional kernel of the map $x \mapsto x \times v$, and then calculate $v \star x$ for x a basis vector other than v for this kernel. Then the next vector is determined by the fact that its leading coefficient is 1 and it lies both in this space and in one of W'_- or W'_+. First we have

$$\begin{aligned} e'_{\overline{\omega}} \star W &= (e_{\overline{\omega}} + e_{-1}) \star \langle e_{-1}, e_{-\omega} - e_0 + e_\omega \rangle \\ &= \langle e_{-1}, e_\omega - e_{\overline{\omega}} \rangle \end{aligned} \tag{6.4}$$

so $e'_\omega = e_\omega - e_{\overline{\omega}} - e_{-1}$. Next we calculate

$$\begin{aligned} e'_\omega \star W &= (e_\omega - e_{\overline{\omega}} - e_{-1}) \star \langle e_{-1}, e_{-\overline{\omega}} + e_{-\omega} - e_0 + e_\omega \rangle \\ &= \langle e_{\overline{\omega}} + e_{-1}, e_{-\omega} - e_0 + e_\omega - e_{\overline{\omega}} \rangle \end{aligned} \tag{6.5}$$

and deduce that $e'_{-\omega} = e_{-\omega} - e_0 + e_\omega + e_{-1}$. Then we have

$$\begin{aligned} e'_{-\omega} \star W &= (e_{-\omega} - e_0 + e_\omega + e_{-1}) \star \langle e_{\overline{\omega}} + e_{-1}, e_1 - e_{-\overline{\omega}} - e_\omega \rangle \\ &= \langle e_\omega - e_{\overline{\omega}} - e_{-1}, e_{-\overline{\omega}} + e_{-\omega} - e_0 + e_{\overline{\omega}} \rangle \end{aligned} \tag{6.6}$$

and therefore $e'_{-\overline{\omega}} = e_{-\overline{\omega}} + e_{-\omega} - e_0 + e_\omega - e_{-1}$, and finally by using the inner products we obtain $e'_1 = e_1 - e_{-\overline{\omega}} - e_\omega - e_{\overline{\omega}} - e_{-1}$.

To summarise, we have shown that $f_{1,0,0}$ is represented with respect to the ordered basis $\{e_{-1}, e_{\overline{\omega}}, e_\omega, e_0, e_{-\omega}, e_{-\overline{\omega}}, e_1\}$ by the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & -1 & 1 & 1 & 0 \\ -1 & -1 & -1 & 0 & 0 & -1 & 1 \end{pmatrix}. \tag{6.7}$$

We must now check that this element preserves the algebraic structure. Checking the inner product is a triviality: the only non-obvious cases to check are $e'_{\overline{\omega}}.e'_1$, $e'_\omega.e'_1$ and $e'_\omega.e_{\overline{\omega}}$. The fact that the basis vectors e'_t lie in the correct eigenspaces W'_+, W'_0 or W'_- means that the cross products with e'_0 are correct. The cyclic symmetry of the trilinear form implies that all values of the trilinear form at triples of basis vectors involving e'_0 are correct. All other triples involve either two vectors from W'_- or two from W'_+, so it is sufficient to check the products of such pairs. We calculate

$$\begin{aligned}
e'_\omega \times e'_{\overline{\omega}} &= (e_\omega - e_{\overline{\omega}} - e_{-1}) \times (e_{\overline{\omega}} + e_{-1}) \\
&= e_\omega \times (e_{\overline{\omega}} + e_{-1}) = e_{-1} \\
e'_{\overline{\omega}} \times e'_1 &= (e_{\overline{\omega}} + e_{-1}) \times (e_1 - e_{-\overline{\omega}} - e_\omega - e_{\overline{\omega}} - e_{-1}) \\
&= e_{\overline{\omega}} \times (e_1 - e_\omega - e_{-\overline{\omega}}) + e_{-1} \times (e_1 - e_{-\overline{\omega}}) \\
&= e_{-\omega} + e_{-1} + e_\omega - e_0 = e'_{-\omega} \\
e'_1 \times e'_\omega &= (e_1 - e_{-\overline{\omega}} - e_\omega - e_{\overline{\omega}} - e_{-1}) \times (e_\omega - e_{\overline{\omega}} - e_{-1}) \\
&= e_{-\overline{\omega}} + e_{-\omega} - e_{-1} + e_\omega - e_0 = e'_{-\overline{\omega}} \\
e'_{-1} \times e'_{-\omega} &= e_{-1} \times (e_{-\omega} - e_0 + e_\omega + e_{-1}) \\
&= e_{\overline{\omega}} + e_{-1} = e'_{\overline{\omega}} \\
e'_{-\overline{\omega}} \times e'_{-1} &= (e_{-\overline{\omega}} + e_{-\omega} - e_0 + e_\omega - e_{-1}) \times e_{-1} \\
&= e_\omega - e_{\overline{\omega}} - e_{-1} = e'_\omega \\
e'_{-\omega} \times e'_{-\overline{\omega}} &= (e_{-\omega} - e_0 + e_\omega + e_{-1}) \times (e_{-\overline{\omega}} + e_{-\omega} - e_0 + e_\omega - e_{-1}) \\
&= (e_{-\omega} - e_0 + e_\omega + e_{-1}) \times (e_{-\overline{\omega}} + e_{-1}) \\
&= e_1 - e_{\overline{\omega}} - e_{-\overline{\omega}} - e_{-1} - e_\omega = e'_1
\end{aligned} \tag{6.8}$$

which concludes the proof that the cross product is invariant. Finally we need to prove that the star product is invariant. We need to check all the fourteen defining equations. This is similarly straightforward, and is left as an exercise for the reader.

In the case when $q > 3$, this element and its conjugates by the maximal torus are sufficient to generate the whole root subgroup, of order q^3. In the case $q = 3$ we need to calculate the case $\beta = 1$, $\alpha = \gamma = 0$ as well. For completeness we give the root elements for the roots $\overline{\omega}$ and -1 here:

$$\begin{pmatrix}
1 \\
0 & 1 \\
-1 & 0 & 1 \\
0 & 1 & 0 & 1 \\
1 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & -1 & 0 & 1 \\
1 & 0 & -1 & 0 & 1 & 0 & 1
\end{pmatrix}, \quad
\begin{pmatrix}
1 \\
0 & 1 \\
0 & 0 & 1 \\
1 & 0 & 0 & 1 \\
0 & -1 & 0 & 0 & 1 \\
1 & 0 & 1 & 0 & 0 & 1 \\
1 & -1 & 0 & -1 & 0 & 0 & 1
\end{pmatrix}. \tag{6.9}$$

6.3. *Root elements on* $\mathbb{W}_{26}$

Again there are three orbits of the Weyl group on roots, namely the inner, middle and outer roots. We shall show that it is only necessary to prove existence of the inner root elements, as the others can be constructed from these. We shall first construct the root element corresponding to the inner root $-i$. This is defined as the unique unitriangular matrix which fixes $e(-1)$ and $e(0)$ and maps $e(\overline{\omega}0) \mapsto e(\overline{\omega}0) + e(-i)$. As before, uniqueness is immediate from our stabilizer theorem (Theorem 4).

Moreover, the proof of Theorem 4 tells us how to calculate the root element, which we shall call $x(-i)$. It acts as

$$\begin{pmatrix}
1 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 \\
1 & 1 & 1 & 0 \\
1 & 0 & 1 & 1
\end{pmatrix}$$

on each of the four 4-spaces $E(\overline{\omega}, \omega^i, -\omega, -\overline{\omega}^i)$, $E(\overline{\omega}^i, \omega, -\omega^i, -\overline{\omega})$, $E(\omega^j, \overline{\omega}^k, -\overline{\omega}^j, -\omega^k)$, $E(\omega^k, \overline{\omega}^j, -\overline{\omega}^k, -\omega^j)$, and as

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}$$

on $E(0, -1, -i, i, 1, \overline{\omega}0)$, as well as the identity on $E(\pm j, \pm k)$. For the remainder of this subsection, write $e'(r)$ for the image of $e(r)$ under $x(-i)$.

We must now show that this element $x(-i)$ preserves the three products. Note first that the blocks of the action are given by the horizontal lines in Fig. 3. Moreover, $x(-i)$ is centralized by the element ρ_1 of the Weyl group, which reflects the picture in the horizontal axis. This involution swaps the 4×4 blocks in pairs, in such a way that the given bases are dual to each other with respect to the inner product B. Thus to show that $x(-i)$ preserves B it suffices to check the fixed block $E(0, -1, -i, i, 1, \overline{\omega}0)$. This is a small and easy calculation.

Next consider the cross product. Let W_{16} denote the space spanned by the 16 coordinate vectors $e(r)$ for $r \in \omega Q_8 \cup \overline{\omega} Q_8$, and let W_{10} denote the space spanned by the other ten.

Theorem 5. *The element $x(-i)$ preserves the cross product on $\mathbb{W}_{26}$.*

Proof. We consider first the case $u \times v$ where $u, v \in W_{10}$. But this product is zero except for the product by $e(\omega 0)$, which acts as an identity on the 8-space $E(\pm 1, \pm i, \pm j, \pm k)$, so this case is trivial.

Next consider the products of $u \in W_{16}$ with $v \in W_{10}$. Since both W_{16} and W_{10} are invariant under the action of $x(-i)$, and since the products of the coordinate vectors in W_{10} with those in W_{16} lie in W_{16}, we know that the only values of the trilinear form which we need to check are $T(u, v, w)$ where $u \in W_{10}$ and $v, w \in W_{16}$. Hence by the symmetry of the trilinear form, we have reduced to the case when $u, v \in W_{16}$.

Now all the products in W_{16} are zero except when the roots lie symmetrically about the vertical axis (when the product can be $e(\pm j)$ or $e(\pm k)$), or about the horizontal axis (giving $e(\pm 1)$ or $e(\pm i)$), or both (giving $e(\omega 0)$ or $e(\overline{\omega}0)$).

First consider the case when r, s are not symmetric about the horizontal axis. Depending on which rows r and s lie in, the products of terms in those rows may be always zero (in which case the result is trivial), or may involve just one of $e(j)$, $e(k)$, $e(-k)$ or $e(-j)$ (in which case we need to check the coefficient of this term). The case when r lies in the first row and s lies in the second is typical, so consider this case. If r and s lie in the same column, then all the cross terms in the expansion of $e'(r) \times e'(s)$ cancel out, and the diagonal terms are all zero, so $e'(r) \times e'(s) = 0 = e(r) \times e(s)$ as required. If r and s lie symmetrically about the vertical axis, then $e(r) \times e(s) = e(j)$, and all the trailing terms in $e'(r) \times e'(s)$ are zero. The only remaining cases which could be non-zero are $r = -\overline{\omega}$ and $s = -\overline{\omega}^j$ or $\overline{\omega}^k$. In both these cases we check that $e'(r) \times e'(s)$ picks up two terms $e(j)$, which cancel out.

Now consider the case when r and s are symmetrically placed about the horizontal axis. We may suppose that r lies in the second row and s lies in the third row, as the case of the first and fourth rows is the same. If r and s lie in the same column, then again all the cross terms in $e'(r) \times e'(s)$ cancel out, and the diagonal terms are all zero, so the result follows. This leaves six cases to consider individually:

$$\begin{aligned} e'(\omega^j) \times e'(\overline{\omega}^j) &= e(\omega^j) \times (e(\omega^k) + e(\overline{\omega}^j)) \\ &= e(-1) \\ e'(\omega^j) \times e'(-\overline{\omega}^k) &= e(\omega^j) \times (e(\omega^k) + e(\overline{\omega}^j) + e(-\overline{\omega}^k)) \end{aligned}$$

$$
\begin{aligned}
&= e(-i) + e(-1) = e'(-i)\\
e'(\omega^j) \times e'(-\omega^j) &= e(\omega^j) \times (e(\omega^k) + e(-\overline{\omega}^k) + e(-\omega^j))\\
&= e(\omega 0) + e(-i) = e'(\omega 0)\\
e'(\overline{\omega}^k) \times e'(-\overline{\omega}^k) &= (e(\omega^j) + e(\overline{\omega}^k)) \times (e(\omega^k) + e(\overline{\omega}^j) + e(-\overline{\omega}^k))\\
&= e(\overline{\omega}0) + e(-i) = e'(\overline{\omega}0)\\
e'(\overline{\omega}^k) \times e'(-\omega^j) &= (e(\omega^j) + e(\overline{\omega}^k)) \times (e(\omega^k) + e(-\overline{\omega}^k) + e(-\omega^j))\\
&= e(i) + e(\omega 0) + e(\overline{\omega}0) + e(-1) + e(-i) = e'(i)\\
e'(-\overline{\omega}^j) \times e'(-\omega^j) &= (e(\omega^j) + e(\overline{\omega}^k) + e(-\overline{\omega}^j)) \times (e(\omega^k) + e(-\overline{\omega}^k) + e(-\omega^j))\\
&= e(1) + e(i) + e(0) + e(\overline{\omega}0) + e(-1) = e'(1).
\end{aligned} \tag{6.10}
$$

This concludes the proof. □

Now we need to prove invariance of the star product under $x(-i)$.

THEOREM 6. *The element $x(-i)$ preserves the star product on $\mathbb{W}_{26}$.*

Proof. First note that the star product of $u \in W_{16}$ with $v \in W_{10}$ is zero everywhere, so we can consider separately the product on W_{10} and on W_{16}. We consider first the product on W_{16}. We have three main cases to consider: the two vectors lie in the same row, or two rows equidistant from the horizontal axis, or two other rows. We do one of each case, as the others are identical. First suppose both vectors lie in the first row, so that the products are

$$
\begin{aligned}
e(\overline{\omega}^i) \star e(-\overline{\omega}) &= e(j)\\
&= e(\omega) \star e(-\omega^i)
\end{aligned} \tag{6.11}
$$

and otherwise zero. Therefore the only cases we need to calculate are

$$
\begin{aligned}
e'(-\overline{\omega}) \star e'(-\omega^i) &= (e(-\overline{\omega}) + e(-\omega^i) + e(\overline{\omega}^i)) \star (e(-\omega^i) + e(\omega) + e(\overline{\omega}^i))\\
&= e(j) + e(j) = 0\\
e'(-\overline{\omega}) \star e'(\omega) &= (e(-\overline{\omega}) + e(-\omega^i) + e(\overline{\omega}^i)) \star (e(\omega) + e(\overline{\omega}^i))\\
&= e(j) + e(j) = 0.
\end{aligned} \tag{6.12}
$$

Next suppose the first vector lies in the first row, and the second in the second. Now the non-zero terms come in four pairs:

$$
\begin{aligned}
e(\overline{\omega}^i) \star e(\overline{\omega}^k) &= e(\omega^j) \star e(\omega) = e(\overline{\omega}^i)\\
e(\overline{\omega}^i) \star e(-\overline{\omega}^j) &= e(\omega^j) \star e(-\omega^i) = e(\omega),\\
e(-\overline{\omega}) \star e(\overline{\omega}^k) &= e(\omega) \star e(-\omega^k) = e(-\omega^i),\\
e(-\overline{\omega}) \star e(-\overline{\omega}^j) &= e(-\omega^i) \star e(-\omega^k) = e(-\overline{\omega}).
\end{aligned} \tag{6.13}
$$

If our two vectors are in the same column, then the cross terms cancel out, and the rest are zero. If our two vectors are equidistant from the vertical axis, then their cross-product is $e(j)$, so rather than a single term $e(r) \star e(s)$ we have to consider a pair of such terms: but then every term in the product cancels out. This leaves just two non-trivial cases to calculate:

$$
\begin{aligned}
e'(\omega) \star e'(-\omega^k) &= (e(\omega) + e(\overline{\omega}^i)) \star (e(-\omega^k) + e(-\overline{\omega}^j) + e(\omega^j))\\
&= e(-\omega^i) + e(\overline{\omega}^i) + e(\omega) = e'(-\omega^i),\\
e'(-\omega^i) \star e'(-\omega^k) &= (e(-\omega^i) + e(\omega) + e(\overline{\omega}^i)) \star (e(-\omega^k) + e(-\overline{\omega}^j) + e(\omega^j))\\
&= e(-\overline{\omega}) + e(-\omega^i) + e(\overline{\omega}^i) = e'(-\overline{\omega}).
\end{aligned} \tag{6.14}
$$

Now suppose the first vector lies in the first row, and the second in the fourth row. In this case, most of the cross products are non-zero, which means we have to consider the star products in pairs, and then it is easy to see that all the terms cancel out. There are also two cases $e'(\omega) \star e'(-\omega) + e'(\omega^i) \star e'(-\omega^i)$ and the same with ω replaced by $\overline{\omega}$, in which again all terms cancel out except the term $e(\omega) \star e(-\omega) + e(\omega^i) \star e(-\omega^i) = e(0)$. The only non-trivial cases left are

$$\begin{aligned}
e'(\omega)\star e'(\omega^i) &= e(-i)+e(-1)\\
&= e'(-i),\\
e'(-\omega)\star e'(-\omega^i) &= e(i)+e(0)+e(-i)+e(-1)\\
&= e'(i),\\
e'(-\overline{\omega})\star e'(-\overline{\omega}^i) &= e(1)+e(i)+e(0)+e(-1)\\
&= e'(1)
\end{aligned}\tag{6.15}$$

in which the calculations are again easy because all the cross-terms cancel out.

Finally we deal with the star product on W_{10}. Products among $e(\pm j)$ and $e(\pm k)$ are trivially fixed by $x(-i)$, and products between these and the rest simply map the row $-1,-i,i,1$ to one of the rows of basis vectors from W_{16}. Since the action on $E(\pm 1,\pm i)$, modulo $E(0)$, is the same as on each of these rows, these instances of the product are also preserved. This just leaves the product on the horizontal axis, which is easy to check. □

The other root elements are obtained by

(1) using the Weyl group to get the elements corresponding to all inner roots,
(2) squaring these to get the elements corresponding to outer roots, and
(3) computing the commutator

$$[x(-j),x(\overline{\omega}^i]x(-1)=x(\omega^k)\tag{6.16}$$

to get the elements corresponding to the middle roots.

In particular we can exhibit an explicit middle root element, such as $x(\omega)$, which acts

(1) as $\begin{pmatrix}1&0\\1&1\end{pmatrix}$ on each of the six 2-spaces

$$E(-1,\overline{\omega}),E(-\overline{\omega},1),E(k,-\overline{\omega}^k),E(\overline{\omega}^k,-k),E(\omega^j,\omega^i),E(-\omega^i,-\omega^j),\tag{6.17}$$

(2) as the identity on $E(\pm\omega^k)$, and
(3) as $\begin{pmatrix}1&0\\1&1\end{pmatrix}\otimes\begin{pmatrix}1&0\\1&1\end{pmatrix}$ on each of the three 4-spaces

$$E(j,-\overline{\omega}^j,i,-\overline{\omega}^i),E(\overline{\omega}^i,-i,\overline{\omega}^j,-j),E(\omega,0,\overline{\omega}0,-\omega).\tag{6.18}$$

Notice that the roots for each of these spaces are aligned on lines parallel to the line joining $\omega,0,-\omega$ in Fig. 5.

7. *Counting points*

The Suzuki ovoid in $\mathbb{W}_4$, the Ree–Tits unital in $\mathbb{W}_7$, and the generalized octagon in $\mathbb{W}_{26}$, may all be defined in the same way.

Definition 5.

(i) *A* point *is a 1-dimensional subspace $\langle v\rangle$ of $\mathbb{W}$ such that $v=v\star w$ for some w.*
(ii) *Two distinct points $\langle u\rangle$ and $\langle v\rangle$ are*
  (a) adjacent *if $u=v\star x$ for some x, and*
  (b) opposite *if $u.v\neq 0$.*
(iii) *A set of $q+1$ mutually adjacent points is called a* line.

It follows immediately from the following theorem that in the cases $\mathbb{W}_4$ and $\mathbb{W}_7$ every pair of points is opposite.

THEOREM 7. *Let $\langle v\rangle$ be a point in $\mathbb{W}$. Then, up to scalar multiplication, the leading term of v is e_r, where r is an outer root. In particular*

(i) *Every point in $\mathbb{W}_4$ or $\mathbb{W}_7$ has leading term e_1 or e_{-1}.*

(ii) *Every point in $\mathbb{W}_{26}$ has leading term e_r where $r = \pm 1, \pm j, \pm\overline{\omega}, \pm\overline{\omega}^i$.*

Proof. By definition v is isotropic, so without loss of generality the leading term of v is e_r for some $r \in U$. The condition $v = v \star w$ implies that e_r has the property that r is a short root and $\phi(r)$ is a long root which is the sum of r and another short root. This means that the inner product of r with $\phi(r)$ is 1, 3/2 or 1 respectively in the three cases, in other words r is an outer root. These have already been classified, and in the case of $\mathcal{G}$ and $\mathcal{E}$ they are just $r = \pm 1$. In the case $\mathcal{H}$ they are $r = \pm 1, \pm j, \pm\overline{\omega}, \pm\overline{\omega}^i$. □

We shall show that every point except $E(-1)$ is in the same orbit under the group as a point with a lower leading term. It follows that every point is in the same orbit as $E(-1)$.

THEOREM 8. *Let $\langle v\rangle$ be a point in $\mathbb{W}_4$, with leading term e_1. Then there is an element of the automorphism group of $\mathbb{W}_4$ which maps v to e_{-1}.*

Proof. Applying elements of the root group as necessary to remove the terms in e_i and e_{-i} from v, we may assume that $v = e_1 + \lambda e_{-1}$, and that w has leading term e_i. Since $e_1 \star e_{-i} = e_i$, it follows that w has no term in e_{-i}. Since $v.w = 0$, it follows that w has no term in e_{-1}. Finally, since $e_{-1} \star e_i = e_{-i}$, it follows that $\lambda = 0$. Therefore $v = e_1$, which is mapped to e_{-1} by an element of the Weyl group. □

As an immediate corollary, we have that the number of points is $q^2 + 1$, and the group acts 2-transitively on the points. Since $E(1)$ is not adjacent to $E(-1)$, it follows that no pair of points is adjacent, and hence there are no lines in $\mathbb{W}_4$.

THEOREM 9. *Let $\langle v\rangle$ be a point in $\mathbb{W}_7$, with leading term e_1. Then there is an element of the automorphism group of $\mathbb{W}_7$ which maps v to e_{-1}.*

Proof. Applying elements of the root group as necessary, we may assume that v has no term in $e_{-\overline{\omega}}$, $e_{-\omega}$ or e_0. Since v is isotropic, it has no term in e_{-1}. Therefore the element $e_t \mapsto e_{-t}$ of the Weyl group maps v to a vector with no term in e_1. Since this is a point, its leading term is e_{-1}, so v must have been e_1. □

Again, it follows immediately that the number of points is $q^3 + 1$, and that the group acts 2-transitively on the points. No pair of points is adjacent, and there are no lines in $\mathbb{W}_7$.

THEOREM 10. *Let $\langle v\rangle$ be a point in $\mathbb{W}_{26}$, other than $E(-1)$. Then there is an element of the automorphism group of $\mathbb{W}_{26}$ which maps v to w, where the leading term of w is strictly lower than the leading term of v.*

Proof. The possible leading terms of v are, in increasing order

$$e(-1), e(\overline{\omega}), e(\overline{\omega}^i), e(-j), e(j), e(-\overline{\omega}^i), e(-\overline{\omega}), e(1).$$

Now ρ_1 acts on these vectors as the permutation $(\overline{\omega}, \overline{\omega}^i)(-j, j)(-\overline{\omega}^i, -\overline{\omega})$, and ρ_2 acts as $(-1, \overline{\omega})(\overline{\omega}^i, -j)(j, -\overline{\omega}^i)(-\overline{\omega}, 1)$. In each case, therefore, one of the elements ρ_1 or ρ_2 takes the given leading term to the next one down in the sequence, while fixing the zero coefficients in v corresponding to the higher terms in the sequence. Therefore, all that remains is to prove that suitable root elements can be used to remove the term in v corresponding to the next term down in this sequence.

For example, if v has leading term $e(1)$, we can remove the term in $e(-\overline{\omega})$ from v by using a conjugate of $x(\omega)$ by a suitable element of the torus. Then ρ_2 conjugates the resulting vector to one with a lower leading term, namely $e(-\overline{\omega})$. The same argument deals with the case when the leading term is $e(\overline{\omega})$.

The other cases are only slightly more difficult. If the leading term is $e(-\overline{\omega})$ or $e(\overline{\omega}^i)$ then we have the standard generators of $\mathrm{Aut}(\mathbb{W}_4)$ acting on the appropriate 4-space $E(-\overline{\omega}^i, -\omega^j, -\omega^k, -\overline{\omega})$ or $E(\overline{\omega}, \omega^k, \omega^j, \overline{\omega}^i)$, with the root element $e(-k)$ acting faithfully. As the argument is the same in both cases we give the latter. We may use the root group to remove the terms in $e(\omega^j)$ and $e(\omega^k)$. Now the leading term of w is $e(\overline{\omega}^k)$, and $e(\overline{\omega}^k) \star e(\overline{\omega}) = e(\omega^k)$, so the term in $e(\overline{\omega})$ in v must be zero, as required. The same argument deals with the case when the leading term is $e(j)$, with $\mathrm{Aut}(\mathbb{W}_4)$ acting on the 4-space $E(-j, -k, k, j)$ modulo $E(0)$.

The remaining two cases are $e(-j)$ and $e(-\overline{\omega}^i)$. In these two cases we use $x(\omega)$ again, acting on the 4-space $E(\overline{\omega}^i, -i, \overline{\omega}^j, -j)$ or $E(j, -\overline{\omega}^j, i, -\overline{\omega}^i)$ as appropriate. Consider the first case, as the other is identical. The leading term of v is $e(-j)$ and the leading term of w is $e(-k)$. We use the root group to remove the term in $e(\overline{\omega}^j)$ from v. But now $e(-k) \star e(-i) = e(\overline{\omega}^j)$, so the term in $e(-i)$ must also be zero. Finally consider the term in $e(\overline{\omega}^i)$. We must have $v \times w = 0$, but $e(-k) \times e(\overline{\omega}) = e(\omega^j)$, which cannot be cancelled out by any lower term of $v \times w$, so the term in $e(\overline{\omega}^i)$ in v is also zero, as required. □

Just as in $\mathbb{W}_4$ and $\mathbb{W}_7$, this argument also allows us to count the points. The root groups used to remove the next term are alternately of order q and q^2. Thus as we go up the sequence, the number of points with given leading term is multiplied alternately by q and then by q^2. Therefore the total number of points is

$$1 + q + q^3 + q^4 + q^6 + q^7 + q^9 + q^{10} = (1 + q)(1 + q^3)(1 + q^6).$$

This time, each point is opposite to precisely q^{10} other points. Moreover, the group is transitive on pairs of opposite points. Each point is adjacent to exactly $q + q^3$ points. Moreover, if $\langle v\rangle$ and $\langle w\rangle$ are adjacent, then for every $\lambda \neq 0$ the 1-space $\langle v + \lambda w\rangle$ is a point adjacent to both of them. Thus we obtain a set of $q + 1$ mutually adjacent points, which is a line.

8. *Properties of the Suzuki and Ree groups*

8.1. *The group orders*

Since in each case the group acts transitively on pairs of opposite points, it suffices, in order to calculate the group order, to calculate the stabilizer of any pair of opposite points, say $E(1)$ and $E(-1)$.

THEOREM 11.
(i) *The stabilizer of two opposite points in* $\mathbb{W}_4$ *is* C_{q-1}.
(ii) *The stabilizer of two opposite points in* $\mathbb{W}_7$ *is* C_{q-1}.
(iii) *The stabilizer of two opposite points in* $\mathbb{W}_{26}$ *has shape* $C_{q-1} \times \mathrm{Aut}(\mathbb{W}_4)$ *and has order* $q^2(q^2 + 1)(q - 1)^2$.

Proof.
(i) If the two points $E(1)$ and $E(-1)$ are fixed, then so are

$$\begin{aligned}(E(1) \star W) \cap E(-1)^\perp &= E(1, i) \cap E(-1)^\perp \\ &= E(i)\end{aligned}$$

$$E(-1) \star E(i) = E(-i) \tag{8.1}$$

Therefore the stabilizer consists of diagonal matrices, so by Theorem 1 is cyclic of order $q-1$.

(ii) If the two points $E(1)$ and $E(-1)$ are fixed, then so is $E(1) \times E(-1) = E(0)$, and the result follows from Theorem 3.

(iii) Both the root element $x(-k)$ and the Weyl group element ρ_1 are in the stabilizer the opposite points $E(-1)$ and $E(1)$. Moreover, these elements together with the torus act on the 4-space $E(\overline{\omega}, \omega^k, \omega^j, \overline{\omega}^i)$ as the generators of $\mathrm{Aut}(\mathbb{W}_4)$ do on $\mathbb{W}_4$. Hence, by multiplying by a suitable element of $\mathrm{Aut}(\mathbb{W}_4)$, we may assume that $E(\overline{\omega})$, $E(\omega^k)$, $E(\omega^j)$ and $E(\overline{\omega}^i)$ are all fixed. Therefore so are

$$\begin{aligned} E(1) \times E(\overline{\omega}^i) &= E(-\omega^i), \\ E(-\omega^i) \star E(\omega^j) &= E(\omega), \\ E(\omega) \times E(1) &= E(-\overline{\omega}), \\ E(\overline{\omega}) \times E(-\overline{\omega}) &= E(\overline{\omega}0), \end{aligned} \tag{8.2}$$

and Theorem 4 implies that our symmetry is diagonal. A subgroup C_{q-1} of the diagonal group $C_{q-1} \times C_{q-1}$ already lies inside $\mathrm{Aut}(\mathbb{W}_4)$, so the result follows.

□

Corollary 1.

(i) *The order of the automorphism group of* $\mathbb{W}_4$ *is* $q^2(q^2+1)(q-1)$.

(ii) *The order of the automorphism group of* $\mathbb{W}_7$ *is* $q^3(q^3+1)(q-1)$.

(iii) *The order of the automorphism group of* $\mathbb{W}_{26}$ *is*

$$(1+q)(1+q^3)(1+q^6)q^{10}(1+q^2)q^2(q-1)^2.$$

8.2. *Simplicity when $q > p$*

In order to prove simplicity we first need information on the point stabilizers. Since we now have the group orders, we also have the orders of the point stabilizers. Obtaining the precise structures is a matter of calculation with root elements, the torus and the Weyl group.

Theorem 12.

(i) *The point stabilizer in* $\mathbb{W}_4$ *is a soluble group of lower triangular matrices, of order* $q^2(q-1)$. *The stabilizer of a pair of points is a dihedral group of order* $2(q-1)$.

(ii) *The point stabilizer in* $\mathbb{W}_7$ *is a soluble group of lower triangular matrices, of order* $q^3(q-1)$. *The stabilizer of a pair of points is a dihedral group of order* $2(q-1)$.

(iii) *The stabilizer of a point in* $\mathbb{W}_{26}$ *is a group of shape* $q.q^4.q.q^4.(C_{q-1} \times \mathrm{Aut}(\mathbb{W}_4))$. *The stabilizer of a pair of points is* $D_{2(q-1)} \times \mathrm{Aut}(\mathbb{W}_4)$.

Now we are ready to prove simplicity. The following result is well-known, and easy to prove.

Lemma 1. *If G is a primitive permutation group and the point stabilizer is soluble, then G is simple if it is perfect.*

Proof. If K is any proper non-trivial normal subgroup of G, and H is a point stabilizer in G, then $G = HK$ by maximality, whence $G/K = HK/K \cong H/(H \cap K)$ is soluble, contradicting the assumption that G is perfect. □

Theorem 13. $\mathrm{Aut}(\mathbb{W}_4)$ *is simple if* $q > 2$.

Proof. If $q > 2$, then the point stabilizer in $\mathrm{Aut}(\mathbb{W}_4)$ is generated by conjugates of the diagonal elements, of order $q-1$. Hence $\mathrm{Aut}(\mathbb{W}_4)$ itself is generated by conjugates of these elements. Since these elements lie in the dihedral group $D_{2(q-1)}$, and $q-1$ is odd, they are commutators, and therefore $\mathrm{Aut}(\mathbb{W}_4)$ is perfect. Hence it is simple, for $q > 2$. □

THEOREM 14. $\mathrm{Aut}(\mathbb{W}_7)$ *is simple if* $q > 3$.

Proof. If $q > 3$, then the point stabilizer in $\mathrm{Aut}(\mathbb{W}_7)$ is generated by conjugates of the diagonal elements of order $q-1$. Now all involutions are commutators, since they lie in $2^3{:}7{:}3$, and the elements of order $(q-1)/2$ are commutators since they lie in a dihedral group D_{q-1} of twice odd order. Hence $\mathrm{Aut}(\mathbb{W}_7)$ is perfect, and therefore simple, for $q > 3$. □

In the case of $\mathbb{W}_{26}$ we need the following stronger result, usually known as Iwasawa's Lemma.

LEMMA 2. *If G is a finite primitive permutation group, and the point stabilizer H has a normal soluble subgroup S whose G-conjugates generate G, then G is simple if it is perfect.*

Proof. If K is a proper non-trivial normal subgroup of G, then K is transitive, so $HK = G$. Now G is generated by the conjugates of S, and writing $g = hk$ with $h \in H$, $k \in K$, for an arbitrary element $g \in G$, we have $S^g = S^{hk} = S^k \leq SK$ by normality of K. Hence $SK = G$, and $G/K = SK/K = S/(S \cap K)$ is soluble, contradicting the assumption that G is perfect. □

THEOREM 15. $\mathrm{Aut}(\mathbb{W}_{26})$ *is simple if* $q > 2$.

Proof. If $q > 2$, then the point stabilizer in $\mathrm{Aut}(\mathbb{W}_{26})$ is again generated by conjugates of the diagonal elements of order $q-1$. Moreover, the normalizer of this torus has shape ${C_{q-1}}^2{:}D_{16}$, and therefore these diagonal elements lie in the derived subgroup. Hence $\mathrm{Aut}(\mathbb{W}_{26})$ is perfect. Moreover, the whole torus ${C_{q-1}}^2$ is generated by conjugates of any single non-trivial element. Hence $\mathrm{Aut}(\mathbb{W}_{26})$ is generated by conjugates of the normal soluble subgroup $[q^{10}].C_{q-1}$ of the point stabilizer. It only remains to show that the group acts primitively on the points, and then we apply Iwasawa's Lemma to deduce that $\mathrm{Aut}(\mathbb{W}_{26})$ is simple whenever $q > 2$. Now the point stabilizer has orbits of lengths 1, $q+q^3$, q^4+q^6, q^7+q^9 and q^{10}. For each of the non-trivial suborbits, there is an element of the Weyl group swapping the fixed point with a point in that suborbit. In two of the four cases, this element fuses all suborbits. In the other two, we obtain two orbits, of lengths $1+q^4+q^6+q^{10}$ and $q+q^3+q^7+q^9$. Again, the putative blocks are more than half the size of the whole orbit, so this is impossible, and therefore the group is primitive, as required. □

8.3. *The case $q = p$*

First consider the group $\mathrm{Aut}(\mathbb{W}_4)$ in the case $q = 2$. We have shown that this is a 2-transitive group of order 20 acting on a set of 5 points. Therefore it is the Frobenius group of this order, and may be generated by the permutations $(1, 2, 3, 4, 5)$ and $(2, 3, 5, 4)$.

Next consider the group $\mathrm{Aut}(\mathbb{W}_7)$ in the case $q = 3$.

THEOREM 16. *If $q = 3$ then* $\mathrm{Aut}(\mathbb{W}_7) \cong \mathrm{PSL}_2(8){:}3$.

Proof. We have shown that this is a 2-transitive group of order $28.27.2 = 1512$ acting on a set of 28 points. We have also shown that the Sylow 2-subgroup is elementary abelian of order 8, and has normalizer $2^3{:}7{:}3$ of order 168. Therefore $\mathrm{Aut}(\mathbb{W}_7)$ has a transitive action on the 9 Sylow 2-subgroups. Indeed, Sylow's theorems imply that this action is 3-transitive and faithful.

Hence $\mathrm{Aut}(\mathbb{W}_7)$ embeds in S_9 and is easily seen to be isomorphic to $\mathrm{PSL}_2(8){:}3$. To prove this, we may label the nine points $*, \infty, 0, 1, 2, 3, 4, 5, 6$ and generate the point stabilizer with the permutations $(\infty, 0)(1, 3)(2, 6)(4, 5)$, $(0, 1, 2, 3, 4, 5, 6)$ and $(1, 2, 4)(3, 6, 5)$. Now the stabilizer of a pair of points contains 7:3 to index 2, and since the involutions in $\mathrm{Aut}(\mathbb{W}_7)$ are all conjugate, the extra element may be taken to be $(*, \infty)(1, 6)(2, 5)(3, 4)$. Relabelling the points by the more usual notation for the projective line of order 8, that is ∞ for $*$, 0 for ∞, and η^t for $t = 0, 1, 2, 3, 4, 5, 6$, where $\eta^3 + \eta + 1 = 0$, our generators become the elements $z \mapsto z + 1$, $z \mapsto \eta z$, $z \mapsto z^2$ and $z \mapsto z^{-1}$ which generate the full automorphism group of the projective line, that is $\mathrm{PSL}_2(8){:}3$. Since the latter group also has order 1512, we obtain the isomorphism $\mathrm{Aut}(\mathbb{W}_7) \cong \mathrm{PSL}_2(8){:}3$ as required. □

Finally consider $\mathrm{Aut}(\mathbb{W}_{26})$ in the case $p = 2$.

THEOREM 17. *If $q = 2$, then $\mathrm{Aut}(\mathbb{W}_{26})$ has a subgroup of index 2, which is simple.*

Proof. This can be proved by an application of the transfer map to the Sylow 2-subgroup (Borel subgroup) or to one of the maximal 2-local subgroups (maximal parabolic subgroups) already constructed. For example, consider the stabilizer of the point $E(-1)$, which, since $q = 2$, is the same as the stabilizer of the vector $e(-1)$, and the same as the centralizer of $x(-1)$. Since $x(-1)$ is the square of $x(-i)$, it is straightforward to calculate the centralizer of $x(-i)$, and we find that it has shape $4 \times 2^4.5.4$. Moreover, we see at least three conjugacy classes of inner root elements in the point stabilizer, namely the classes containing $x(-i)$, $x(\overline{\omega}^j)$ and $x(-k)$.

Conversely, the root element $x(-i)$ fixes exactly 31 points. To prove this, note first that the leading term of any fixed point must be one of $e(-1)$, $e(\overline{\omega})$, $e(\overline{\omega}^i)$, $e(-j)$ or $e(j)$. Now there is just one point with leading term $e(-1)$, two with leading term $e(\overline{\omega})$ and eight with leading term $e(\overline{\omega}^i)$. The last two are fused into a single orbit of length 10 under the centralizer of $x(-i)$. Similarly, this centralizer maps the points with leading term $e(j)$ to those with leading term $e(-j)$, so it suffices to consider the latter. In total there are 16 such points, and precisely four of these are fixed by $x(-i)$, namely the images of $e(-j)$ under the group generated by $x(\overline{\omega}^i)$ and $x(\overline{\omega})$. In particular, there are exactly three orbits of the centralizer of the inner root group $x(-i)$ on the points fixed by $x(-i)$, so there are exactly three conjugacy classes in the point stabilizer which consist of conjugates in $\mathrm{Aut}(\mathbb{W}_{26})$ of $x(-i)$. But we have already exhibited three, so there are no more.

Now it is not difficult to see that the point stabilizer is generated by the inner root groups it contains. I claim that the subgroup generated by products of an even number of inner root elements has index 2. To prove this, we first calculate some commutators to show that the subgroup generated by the root groups for the roots $\omega, \omega^i, \omega^j, \omega^k, -1, \overline{\omega}^i, \overline{\omega}$ and the products $x(-i)x(\overline{\omega}^j)$, $x(-i)x(\overline{\omega}^k)$ is normal in the point stabilizer. (Actually we only need to calculate the commutators of $x(\omega)$ and $x(-i)x(\overline{\omega}^k)$ with $x(-k)$ and then the rest follow.) Now extend this normal subgroup by ρ_1 and $x(-i)x(-k)$, which generate a group isomorphic to $\mathrm{Aut}(\mathbb{W}_4) \cong 5.4$. This proves the claim.

Now we apply transfer. Specifically, (37.4) in [**1**] shows that the inner root elements lie outside $\mathrm{Aut}(\mathbb{W}_{26})'$. (It may be objected that this is not an elementary argument, but in fact it only relies on the previous two pages of [**1**], which in this instance is elementary and does not rely on any earlier parts of the book.) Obviously therefore the subgroup generated by products of two inner root elements has index exactly 2 in $\mathrm{Aut}(\mathbb{W}_{26})$. We shall show that this subgroup is simple, whence it is equal to the derived group $\mathrm{Aut}(\mathbb{W}_{26})'$. The root elements corresponding to inner roots lie outside the subgroup. The point stabilizer in this subgroup is soluble, and it is easy to see that the action on the $1 + 20 + 80 + 640 + 1024 = 1755$ points is still primitive: for the only possibility would be that some of the given orbits of $\mathrm{Aut}(\mathbb{W}_{26})$ split into two orbits of equal size for the subgroup, but then simple arithmetic rules out any possiblility for a block size. The structure of the point stabilizer as $2.2^4.2^4.5.4$ shows that a subgroup of index 4 thereof is in the derived group, and the rest of the group is generated by $x(-i)x(-k)$, which is conjugate to $x(\overline{\varpi}^k)x(\overline{\varpi}^j)$, which lies inside the derived group also. Hence $\mathrm{Aut}(\mathbb{W}_{26})'$ is perfect, and therefore simple. □

9. *Further remarks*

9.1. *Identification of the groups*

At this stage we have proved that, for $q > p$, the automorphism groups of our $\mathbb{W}$-algebras are simple groups of the same order as the Suzuki and Ree groups, and therefore they are isomorphic to the Suzuki and Ree groups. However, this argument uses some highly non-trivial parts of the Classification Theorem for Finite Simple Groups, and so one may prefer a more concrete proof. Indeed, in the case of the Suzuki groups, it is transparent that our matrices are the same as those obtained by Suzuki. In the case of the small Ree groups, one can also make the identification with the Tits geometry without too much difficulty. In the case of the large Ree groups, one can find a re-labelling of the coordinates which identifies our basis with that used by Howlett, Rylands and Taylor [**8**] in their computation of explicit matrices.

In the case $q = p$, we showed that $\mathrm{Aut}(\mathbb{W})$ has a subgroup of index p. When $\mathbb{W} = \mathbb{W}_{26}$, this is the Tits simple group, and in the other two cases we see the well-known isomorphisms ${}^2B_2(2) \cong 5{:}4$ and ${}^2G_2(3) \cong \mathrm{PSL}_2(8){:}3$.

9.2. *Subgroups of Suzuki groups*

The maximal subgroups of the Suzuki groups were determined by Suzuki [**15**] in his original paper. In this section we sketch a proof, and briefly describe how these maximal subgroups can be related to the geometry of our algebra.

Theorem 18. *(Suzuki) The maximal subgroups of ${}^2B_2(q)$, $q > 2$, are, up to conjugacy,*

(i) $[q^2].(q-1)$;
(ii) $(q-1){:}2$;
(iii) $(q+\sqrt{2q}+1){:}4$;
(iv) $(q-\sqrt{2q}+1){:}4$;
(v) ${}^2B_2(q_0)$, *for* $q = q_0^r$, *r prime,* $q_0 > 2$.

Proof. (Sketch.) We have already shown that the stabilizer of $E(-1)$ is a (maximal, soluble) subgroup of shape $[q^2].(q-1)$, consisting of lower triangular matrices. Now the stabilizer of $E(-i)$ fixes also

$$\begin{aligned}(E(-i)\star W)\cap E(-i)^\perp &= E(-1,i)\cap E(-i)^\perp\\ &= E(-1)\end{aligned}\tag{9.1}$$

so is a non-maximal subgroup of order $q(q-1)$. This accounts for all the

$$(q^2+1)+q(q^2+1) = (q^2+1)(q+1)\tag{9.2}$$

1-dimensional subspaces of $\mathbb{W}_4$.

No two points are perpendicular, so an isotropic 2-space can contain at most one point. But there are $q+1$ isotropic 2-spaces containing each of the q^2+1 points, which accounts for all of the $(q^2+1)(q+1)$ isotropic 2-spaces. In every case, the stabilizer of the 2-space lies inside the point stabilizer.

The non-singular 2-space $E(-1,1)$ contains exactly two points, namely $E(-1)$ and $E(1)$, so its stabilizer is a dihedral group $(q-1){:}2$. This group also fixes the 2-space

$$E(-1,1)^{\perp} = E(-i,i), \tag{9.3}$$

which contains no points, and together these account for all the $(q^2+1)q^2$ non-singular 2-spaces.

Hence any other maximal subgroup is irreducible. If it is local, then it is of shape $A{:}4$, where A is abelian, since an elementary abelian 2^2 does not act regularly on $\mathbb{W}_4$. On the other hand, if it is non-local it can only be a smaller Suzuki group. To see the remaining local maximal subgroups, it is useful to consider the exterior square of $\mathbb{W}_4$. This is a 6-dimensional space, on which the group acts fixing the vector $e_1 \wedge e_{-1} + e_i \wedge e_{-i}$. Factoring by the 1-space spanned by this vector, we obtain a 5-dimensional space on which the group acts. This contains an invariant 4-space, spanned by $e_{\pm 1} \wedge e_{\pm i}$, on which the group acts as the Frobenius twist of its action on $\mathbb{W}_4$ itself.

Now the q^4 1-spaces which lie outside this 4-space fall into two orbits under the action of the dual of the 5-dimensional orthogonal group, of lengths $(q^4 \pm q^2)/2$. The subgroup $D_{2(q-1)}$ must fix such a vector, which must be in the orbit of length $(q^4+q^2)/2$, so this remains a single orbit on restriction to the Suzuki group. The orbit of length $(q^4-q^2)/2$ must split into at least two orbits on restriction to the Suzuki group, since the latter has order not divisible by q^2-1. Since the stabilizers must have order bigger than $2(q-1)$ in each case, it is not difficult to see that there are exactly two orbits, of lengths $\frac{1}{4}q^2(q-1)(q \pm \sqrt{2q}+1)$, and the stabilizer of a vector in one of these orbits has order $4(q \pm \sqrt{2q}+1)$. In each case the group is a Frobenius group $C_{q\pm\sqrt{2q}+1}{:}4$, which is maximal as it is visibly not contained in any of the other subgroups. □

9.3. *Subgroups of the small Ree groups*

The maximal subgroups of the small Ree groups were determined by Kleidman [**10**].

THEOREM 19. *(Kleidman) The maximal subgroups of ${}^2G_2(q)$, $q > 3$, are, up to conjugacy,*

(i) $[q^3].(q-1)$;
(ii) $2 \times \mathrm{PSL}_2(q)$;
(iii) $(2^2 \times D_{(q+1)/2}){:}3$;
(iv) $(q+\sqrt{3q}+1){:}6$;
(v) $(q-\sqrt{3q}+1){:}6$;
(vi) ${}^2G_2(q_0)$*, where* $q = q_0^r$*, r prime.*

We shall not pretend to prove this here, although our arguments make a significant contribution to the proof. We merely indicate how these maximal subgroups relate to the geometry of our algebra.

We begin by classifying the 1-dimensional subspaces. The ambient orthogonal group $\Omega_7(q)$ has just three orbits on 1-spaces, one of $q^5+q^4+q^3+q^2+q+1$ isotropic 1-spaces, and two orbits one 1-spaces consisting of non-isotropic vectors, of lengths $(q^6 \pm q^3)/2$. We know that there are q^3+1 isotropic 1-spaces in the orbit of $E(-1)$ under the Ree group. Now

$$E(\overline{\omega}) \star W = E(-1,\omega), \tag{9.4}$$

which contains a unique point, namely $E(-1)$. Hence $E(\overline{\omega})$ lies in an orbit of size $q(q^3+1)$. Similarly,

$$\begin{aligned}(E(\omega) \star W) \cap E(\omega)^{\perp} &= E(\overline{\omega},-\omega) \cap E(\omega)^{\perp} \\ &= E(\overline{\omega}),\end{aligned} \tag{9.5}$$

so the stabilizer of $E(\omega)$ is again contained in the stabilizer of $E(-1)$, and therefore $E(\omega)$ lies in an orbit of size $q^2(q^3+1)$. This accounts for all the $(q^3+1)(q^2+q+1)$ isotropic 1-spaces.

Next consider the orbit of $E(0)$, and recall from the proof of Theorem 3 that multiplication by e_0 has ± 1-eigenspaces $E(-1,-\omega,-\overline{\omega})$ and $E(1,\omega,\overline{\omega})$. Since each of these contains a unique point, namely $E(-1)$ and $E(1)$ respectively, it follows from Theorem 3 that the stabilizer of $E(0)$ has order at most $2(q-1)$. But the dihedral subgroup generated by the torus and the Weyl group does indeed fix $E(0)$, so this is full stabilizer. This accounts for all $(q^6+q^3)/2$ 1-spaces in this $\Omega_7(q)$-orbit. In particular, the Ree group is transitive on these 1-spaces. The stabilizer is not in fact a maximal subgroup, as we shall show below.

Finally, the orbit of length $(q^6-q^3)/2$ must split into at least two orbits under the action of the Ree group, with each stabilizer having order bigger than q. In fact, it splits into three orbits. One of these has length $q^3(q^2-q+1)(q-1)/6$ and the corresponding stabilizer is a group of shape $(2^2 \times D_{(q+1)/2})$:3 (for a proof that it has this shape, see below). The other two have lengths

$$q^3(q^2-1)(q \pm \sqrt{3q}+1)/6 \tag{9.6}$$

and the stabilizers are Frobenius groups $C_{q+\sqrt{3q}+1}$:6.

The only other $\Omega_7(q)$-orbit of subspaces which has length less than the order of the Ree group (so that there is some hope of a reasonable classification of them) is the orbit on

$$(q^3+1)(q^2+1)(q+1) \tag{9.7}$$

totally isotropic 3-spaces. Now no isotropic subspace can contain more than one point, and the number of isotropic 3-spaces containing $E(-1)$ is $(q^2+1)(q+1)$. Therefore all isotropic 3-spaces contain a unique point.

We conclude with a sketch of the 2-local analysis. Since $q \equiv 3 \bmod 8$, the formula for the group order shows that the Sylow 2-subgroup has order 8, coming from a factor of 2 in $q-1$ and a factor of 4 in q^3+1.

Now the centralizer of the involution which negates $e(\pm 1)$ and $e(\pm\omega)$ and fixes $e(\pm\overline{\omega})$ and $e(0)$ contains the cyclic group of order $q-1$ consisting of the diagonal elements, together with the Weyl group and the root element defined by $e(0) \mapsto e(0)+e(\overline{\omega})$. These together generate at least $\Omega_3(q)$ acting on the 3-space $E(0,\pm\overline{\omega})$. But this involution negates exactly $q+1$ points, and the point stabilizer contains exactly q^2 involutions, so the involution centralizer has order

$$q^3(q-1).\frac{q+1}{q^2} = q(q^2-1) = 2|\Omega_3(q)|. \tag{9.8}$$

Moreover, since we already know the involution centralizer contains $C_2 \times C_2$, it cannot be $\mathrm{SL}_2(q)$, and therefore it is $2 \times \Omega_3(q) \cong 2 \times \mathrm{PSL}_2(q)$. This group properly contains the stabilizer of $E(0)$, which is therefore not maximal.

It follows from the structure of the involution centralizer that the Sylow 2-subgroup is elementary abelian, and the part of its normalizer which lies in the involution centralizer is $2 \times A_4$. Now there is a unique class of involutions in the point stabilizer, and it is easy to see that involutions of all three classes in $2 \times A_4$ fix points: the central involution fixes $E(-1)$, while the involution $e_t \mapsto e_{-t}$ fixes $\langle e_1 - e_{-\omega} - e_0 + e_\omega - e_{-1}\rangle$, and their product fixes $\langle e_1 + e_{\overline{\omega}} + e_{-\omega} + e_0 + e_\omega - e_{-\overline{\omega}} + e_{-1}\rangle$. Hence all these involutions are conjugate, and the Sylow 2-normalizer has shape 2^3:7:3. It is not maximal because it is contained in ${}^2G_2(3) \cong \mathrm{PSL}_2(8)$:3.

Now the involution centralizer in $\mathrm{PSL}_2(q)$, for $q \equiv 3 \bmod 8$, is a dihedral group $D_{q+1} \cong 2 \times D_{(q+1)/2}$, since $(q+1)/4$ is odd. It follows that the normalizer of a 2^2 is a group of shape $(2^2 \times D_{(q+1)/2})$:3.

9.4. *Subgroups of the large Ree groups*

The maximal subgroups of the large Ree groups were determined by Malle [**12**]. In this section we briefly describe how a number of these maximal subgroups can be related to the geometry of our algebra.

THEOREM 20. *(Malle) The maximal subgroups of* ${}^2F_4(q)$, $q > 2$, *are, up to conjugacy,*

(i) $q.q^4.q.q^4{:}({}^2B_2(q) \times C_{q-1})$;
(ii) $q^2.[q^9]{:}\mathrm{GL}_2(q)$;
(iii) $\mathrm{SU}_3(q){:}2$;
(iv) $\mathrm{PGU}_3(q){:}2$;
(v) ${}^2B_2(q) \wr 2$;
(vi) $\mathrm{Sp}_4(q){:}2$;
(vii) $(C_{q+1} \times C_{q+1}){:}2S_4$;
(viii) $(q + \sqrt{2q} + 1)^2{:}4S_4$;
(ix) $(q - \sqrt{2q} + 1)^2{:}4S_4$, *provided* $q > 8$;
(x) $(q^2 + q\sqrt{2q} + q + \sqrt{2q} + 1){:}12$;
(xi) $(q^2 - q\sqrt{2q} + q - \sqrt{2q} + 1){:}12$;
(xii) ${}^2F_4(q_0)$, *for* $q = q_0^r$, *r prime.*

The Borel subgroup (i.e. the subgroup of lower unitriangular matrices) has order $q^{12}.(q-1)^2$, and is generated by the torus together with the root subgroups corresponding to the negative roots. Adjoining to this the Weyl group element ρ_1 gives the stabilizer of the point $E(-1)$, which has shape $q.q^4.q.q^4.(C_{q-1} \times {}^2B_2(q))$. Adjoining instead the Weyl group element ρ_2 gives the stabilizer of the line $E(-1, \overline{\omega})$, which has shape $[q^{11}]\mathrm{GL}_2(q)$, where $\mathrm{GL}_2(q) \cong C_{q-1} \times \mathrm{SL}_2(q)$ since q is even.

We saw that the stabilizer of two opposite points has shape $C_{q-1} \times {}^2B_2(q)$. Taking the points $E(-1)$ and $E(1)$, this group may be generated by the torus D together with the root elements $x(k)$ and $x(-k)$ corresponding to the roots $\pm k$. Now if we conjugate these root elements by $\rho_2\rho_1\rho_2$ we obtain the root elements corresponding to $\pm i$. It is easy to show that these new root elements commute with the original ones. Therefore we obtain a group ${}^2B_2(q) \times {}^2B_2(q)$ which is normalized by $\rho_2\rho_1\rho_2$ to give a subgroup ${}^2B_2(q) \wr 2$. It is visible that this subgroup preserves the decomposition of the space as $W_{10} \oplus W_{16}$. Indeed, it fixes the 1-space $E(0)$, and is therefore the stabilizer of this 1-space.

We define two lines to be *opposite* if every point on one line is opposite to all but one of the points on the other line. Then the stabilizer of two opposite lines, such as $E(-1, \overline{\omega})$ and $E(-\overline{\omega}, 1)$, has shape $C_{q-1} \times \mathrm{SL}_2(q)$. This group is generated by the torus and the root elements corresponding to the roots $\pm\omega$, and may be extended to $\mathrm{SL}_2(q) \wr 2$ by adjoining $\rho_1\rho_2\rho_1$. In fact, if we adjoin also ρ_1 we obtain a copy of the symplectic group $\mathrm{Sp}_4(q)$ extended by its outer automorphism of order 2. To see this, observe first that this group contains the whole torus and Weyl group, and is generated by these together with any middle root element. In particular, we see that all the generating elements fix $E(\omega 0)$. Modulo this there are two 4-spaces $E(\pm\omega, \pm\omega^k)$ and $E(\pm\omega^i, \pm\omega^j)$ on which the generators act as the symplectic group, swapped by the outer automorphism.

Inside $\mathrm{SL}_2(q) \times \mathrm{SL}_2(q)$ there is a subgroup $(q+1)^2$ which fixes a 2-space. Indeed, this 2-space is determined as the fixed space of a subgroup 3^2, generated by $\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$ in each $\mathrm{SL}_2(q)$, say. In the case when the two factors are generated by $x(\pm i)$ and $x(\pm k)$, the corresponding 2-space is

$$\langle e(\omega 0), e(\omega^k) + e(\omega) + e(0) + e(-\omega) + e(-\omega^k)\rangle. \tag{9.9}$$

Now there is an S_3 acting on this 2-space, so that its full stabilizer is $(q + 1)^2{:}2S_4$.

Similarly, inside ${}^2B_2(q) \wr 2$ there are subgroups $((q \pm \sqrt{2q} + 1){:}4) \wr 2$ which fix 2-dimensional subspaces, and the full stabilizer of such a 2-space is $(q \pm \sqrt{2q} + 1)^2{:}4S_4$. One of these two groups contains a 5^2, which can be defined over the prime field by taking the square of the root element, times the involution in the Weyl group. In the two copies of ${}^2B_2(q)$, such an element acts as

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix} \tag{9.10}$$

on $E(0, -1, -i, i, 1, \overline{\omega}0)$ or $E(0, -j, -k, k, j, \overline{\omega}0)$. Hence the 2-space is

$$\langle e(0), e(-i) + e(-k) + e(\overline{\omega}0) + e(k) + e(i)\rangle. \tag{9.11}$$

9.5. *Infinite fields*

Some of the above, but not all, makes sense over infinite fields. Suppose first that we have an infinite perfect field with a Tits automorphism. In this case, our definitions of the $\mathbb{W}$-algebras go through. The Weyl group is still the same, and so is the torus, although we do need to check generation. For example, in the Suzuki groups we had the torus generated by diagonal elements with entries $(\lambda, \lambda^{\tau-1}, \lambda^{1-\tau}, \lambda^{-1})$, but since $(\tau - 1)(\tau + 1) = \tau^2 - 1 = 1$ this torus element can equally well be written with entries $(\mu^{\tau+1}, \mu, \mu^{-1}, \mu^{-\tau-1})$.

Now, since the multiplicative group of the field is no longer cyclic, it is not obvious that the root group is generated by a single root together with the torus. This has to be checked separately. For example we check the bottom right-hand corner of the Suzuki root group by computing

$$\begin{pmatrix} \lambda^{\tau-1} & 0 \\ 0 & \lambda \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \lambda^{1-\tau} & 0 \\ 0 & \lambda^{-1} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \lambda^{2-\tau} & 1 \end{pmatrix} \tag{9.12}$$

and observe that $\lambda^{2-\tau}$ is an arbitrary element of the field since raising it to the power $\tau+1$ gives $\lambda^{-\tau^2+\tau+2} = \lambda^\tau$, and we are assuming that τ is an automorphism, not just an endomorphism. A similar calculation shows that we also get the whole field in the case of the squares of these root elements. Notice however that this calculation depends crucially on having a perfect field, and even though the generators can be written down over an arbitary field with a Tits endomorphism, this result is probably not true in the general case.

Similarly the classification of points goes through, and the above calculations give 2-transitivity on the points in the case of the Suzuki groups and small Ree groups.

The commutators in the normalizer of the torus of the Suzuki group are now diagonal of shape $(\lambda^2, \lambda^{2(\tau-1)}, \lambda^{2(1-\tau)}, \lambda^{-2})$, which covers all elements of the torus, since the field is perfect. Hence we have the required generation result, and the proof of simplicity goes through. The same is true for the small Ree groups. In the case of the large Ree groups, however, we cannot use Iwasawa's Lemma directly, as it only applies to finite groups.

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