The maximal subgroups of the Baby Monster, I

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published in J. Algebra 211 (1999),1–14

Abstract

In this paper we describe the completion of the determination of the maximal subgroups of the Baby Monster simple group. Our results are proved using computer calculations with matrix generators for the group. The full details of the calculations will appear elsewhere.

1 Introduction

This paper is a sequel to [8], [10], [12], in which a partial classification of the maximal subgroups of the Baby Monster simple group B of Fischer (see [2]) was achieved. The cases left open after [12] were the 2-local subgroups, and the non-local subgroups whose socle is a simple group, isomorphic to one of the following 16 groups.

$$A_6, L_2(11), L_2(16), L_2(17), L_2(19), L_2(23), L_2(25), L_3(3), U_3(3),$$

$$L_3(4), U_3(8), U_4(2), {}^2F_4(2)', G_2(3), M_{11} \text{ and } M_{22}.$$

The maximal 2-local subgroups have recently been completely classified by Meierfrankenfeld and Shpektorov, with a very beautiful argument [4]. In this paper we deal with the remaining non-local subgroups, using computer calculations with the matrix generators for B constructed in [9]. Here we give the theoretical parts of the argument, together with brief descriptions of the computations and results. We make much use of the information in the Atlas [1], including the character table computed by David Hunt [3]. Full details of the calculations will appear elsewhere [13].

Acknowledgements. The calculations described in this paper were begun on a SUN SPARCstation ELC, provided by the School of Mathematics and Statistics of the University of Birmingham with assistance from a grant from the Science and Engineering Research Council Computational Science Initiative. They were completed on a DEC Alpha AXP workstation provided from another SERC CSI grant. I am grateful to both institutions for their financial assistance.

2 Statement of results

We prove the following theorem.

Theorem 2.1 *The maximal subgroups of B are the conjugates of*

1. *the 22 maximal subgroups listed in the ATLAS;*
2. *the following 8 new maximal subgroups:*
   - *(a)* $(S_6 \times L_3(4){:}2).2$
   - *(b)* $(S_6 \times S_6){:}4$
   - *(c)* $L_2(31)$
   - *(d)* $L_2(49){\cdot}2$
   - *(e)* $L_2(11){:}2$
   - *(f)* M_{11}
   - *(g)* $L_2(17){:}2$
   - *(h)* $L_3(3)$

This may be re-stated as follows.

Theorem 2.2 *There are just 30 conjugacy classes of maximal subgroups of B, as listed in Table 1.*

Table 1: The maximal subgroups of the Baby Monster

Group	Order
$2^{\cdot 2}E_6(2){:}2$	306 129 918 735 099 415 756 800
$2^{1+22}{\cdot}Co_2$	354 883 595 661 213 696 000
Fi_{23}	4 089 470 473 293 004 800
$2^{9+16}S_8(2)$	1 589 728 887 019 929 600
Th	90 745 943 887 872 000
$(2^2 \times F_4(2)){:}2$	26 489 012 826 931 200
$2^{2+10+20}(M_{22}{:}2 \times S_3)$	22 858 846 741 463 040
$2^{5+5+10+10}L_5(2)$	10 736 731 045 232 640
$S_3 \times Fi_{22}{:}2$	774 741 019 852 800
$2^{[35]}(S_5 \times L_3(2))$	692 692 325 498 880
$HN{:}2$	546 061 824 000 000
$O_8^+(3){:}S_4$	118 852 315 545 600
$3^{1+8}{:}2^{1+6}{\cdot}U_4(2).2$	130 606 940 160
$(3^2{:}D_8 \times U_4(3).2^2).2$	1 881 169 920
$5{:}4 \times HS{:}2$	1 774 080 000
$S_4 \times {}^2F_4(2)$	862 617 600
$3^{2+3+6}(S_4 \times 2S_4)$	204 073 344
$S_5 \times M_{22}{:}2$	106 444 800
$(S_6 \times L_3(4){:}2).2$	58 060 800
$5^3{\cdot}L_3(5)$	46 500 000
$5^{1+4}{:}2^{1+4}A_5.4$	24 000 000
$(S_6 \times S_6).4$	2 073 600
$5^2{:}4S_4 \times S_5$	288 000
$L_2(49){\cdot}2$	117 600
$L_2(31)$	14 880
M_{11}	7 920
$L_3(3)$	5 616
$L_2(17){:}2$	4 896
$L_2(11){:}2$	1 320
$47{:}23$	1 081

3 General strategy

We work our way up from small subgroups to large ones. Suppose we wish to classify up to conjugacy all subgroups of B isomorphic to a particular simple group S. Then we choose some way of generating S by subgroups H and K with $L \leq H \cap K$. Usually we take $L = H \cap K$, and often L is a normal subgroup of K, but these are computational conveniences and not always necessary. Our aim is then to classify up to conjugacy all occurrences of such an amalgam $(H, K)_L$ in B, and identify which of these generate subgroups isomorphic to S.

Typically our strategy consists of finding first H (inside some known proper subgroup), then L (inside H—this is usually easy), and then $N_B(L)$ (this is much harder), and finally searching through all possibilities for K inside $N_B(L)$.

We describe first the cases where we build up from an A_5: in section 4 we consider the four 'small' cases A_6, $L_2(11)$, $L_2(19)$ and $L_3(4)$, and in section 5 we consider the six 'large' cases $L_2(16)$, $L_2(25)$, $U_4(2)$, ${}^2F_4(2)$, M_{11} and M_{22}. Then we consider the progressively more difficult cases $L_2(23)$, $G_2(3)$, $U_3(8)$, $L_3(3)$, $U_3(3)$ and $L_2(17)$.

4 Groups generated by an (A_5, A_5) amalgam

Many of the groups in the above list can be generated by two subgroups A_5 intersecting in D_{10}. This includes the groups A_6, $L_2(11)$, $L_2(19)$, and $L_3(4)$. It also includes $L_2(31)$ and $L_2(49)$, and thus our present work provides another proof of the results in [10]. In all cases, by the results of [12], we are only interested in groups containing $5B$-elements. We show the following.

Proposition 4.1 *If H is a subgroup of B containing $5B$-elements, and H is isomorphic to A_6, $L_2(11)$, $L_2(19)$ or $L_3(4)$, then H is conjugate to one of the following five groups.*

1. *$H_1 \cong A_6$, with $N(H) \cong M_{10} < Th$,*
2. *$H_2 \cong A_6$, with $N(H) \cong A_6{}^{\cdot}2^2 < (S_6 \times S_6).4$,*
3. *$H_3 \cong A_6$, with $N(H) \cong M_{10}$ contained in a 2-local subgroup,*
4. *$H_4 \cong L_2(11)$, with $N(H) \cong L_2(11){:}2$ maximal,*

5. $H_5 \cong L_2(19)$*, with* $N(H) \cong L_2(19){:}2 < Th$*.*

Now there are two classes of A_5 containing $5B$-elements, and any such A_5 has normalizer S_5. They can be distinguished in various ways. One type of S_5 is contained in the Thompson group, and therefore all its involutions are of class $2D$ in B. The other type has transpositions in class $2C$ (this fact was noted by Norton [5], and we have also verified it computationally), and the A_5 can be found in $L_2(31)$ for example (although we cannot prove this until the end of the present section!). We therefore have three cases to consider, according as the two groups A_5 are both of one type, both of the other type, or one of each type.

The case when both A_5s are of Th-type. Our plan is to find such an A_5, and then find the normalizer in B of a D_{10} in the A_5. This group has the shape $(D_{10} \times 5{:}4 \times 5{:}4){\cdot}2$. The possibilities for the second A_5 are then exactly the 400 conjugates of the first A_5 by elements of the D_{10}-centralizer $5{:}4 \times 5{:}4$.

The results of this analysis are that we obtain one class each of A_6, $L_2(11)$, $L_2(19)$, and $L_2(49)$. Each has index 2 in its normalizer. The groups $A_6{\cdot}2 \cong M_{10}$ and $L_2(19){:}2$ that we obtain are both contained in Th, while the groups $L_2(11){:}2$ and $L_2(49){\cdot}2$ will both turn out to be maximal.

Finding the other type of A_5. We have observed that the other type of A_5 is contained in $L_2(31)$. For later convenience we want this to have a D_{10} in common with the first A_5. We therefore seek first an $L_2(31)$ containing the particular $5B$-element which we used earlier. To begin with, we find $31{:}3$ in Th, normalized by this 5-element. Then we look in the normalizer of the 15-element to find an involution extending to $L_2(31)$. Finally we conjugate the 31-element by a suitable element to ensure that $L_2(31)$ has a D_{10} (not just a 5) in common with the original A_5.

The remaining cases. When both A_5s are of the second type, we find that such an A_5 extends to just three groups A_6, and not to any of the other groups on the list.

One of these groups extends to S_6, and therefore has normalizer $A_6{\cdot}2^2$ in the Baby Monster. This is contained as a diagonal subgroup in the maximal subgroup $(S_6 \times S_6).4$. The other two groups are interchanged by a known element, and each has normalizer $A_6{\cdot}2 \cong M_{10}$. It can easily be shown that there is a proper subgroup (actually contained in a 2-local subgroup) containing both of these groups M_{10}. In particular, such an A_6-normalizer is not maximal.

In the mixed case, we find nothing except the single class of maximal subgroups $L_2(31)$.

5 Other groups containing A_5

The groups $L_2(16)$, $L_2(25)$, $U_4(2)$, ${}^2F_4(2)'$, M_{11} and M_{22} all contain A_5 and can now be classified fairly easily given the work we have already done. The case $U_4(2)$ is easy, since $U_4(2)$ contains an S_6, which in turn contains an S_5 all of whose involutions are in the same $U_4(2)$-class. It was noted by Norton (see [12]) that the only $5B$-type S_5 with all involutions conjugate in B is the one in Th. But we have already seen that this A_5 extends to exactly two (conjugate) groups A_6, each with normalizer M_{10} (contained in Th). In particular, the S_5 does not extend to S_6, so *a fortiori* does not extend to $U_4(2)$.

Similarly, the Tits group ${}^2F_4(2)'$ contains an S_6 all of whose involutions are conjugate, so for the same reason there is no $5B$-type ${}^2F_4(2)'$ in B. (Note that this will also follow from the non-existence of a $5B$-type $L_2(25)$, which is demonstrated below.) There is no $5B$-type M_{22} in B since there is no $5B$-type $L_3(4)$, whereas M_{22} contains a subgroup $L_3(4)$.

The remaining cases were dealt with by more computations. First, to classify subgroups M_{11}, take $L_2(11)$ and extend A_5 to S_5. There is a unique such group, which can easily be verified to be M_{11}.

Subgroups isomorphic to $L_2(16)$. The group $L_2(16)$ may be generated from A_5 by extending D_{10} to D_{30}. Since the centralizer of the D_{10} has order 400, while the centralizer of the D_{30} has order 2, there are just 200 possibilities for the extending element of order 3, falling into 100 inverse pairs. If the A_5 we start with is the subgroup of Th produced above, then it turns out that none of the 100 extensions of this type is $L_2(16)$.

If we start with the other A_5, then all but four of the 100 extensions are easily shown not to be $L_2(16)$. Now extending the A_5 to S_5 cannot normalize any $L_2(16)$, since $L_2(16){:}2$ does not contain S_5. Therefore every

$5B$-type $L_2(16)$ in B is self-normalizing, and occurs twice in our list. In particular, there are at most two classes of such subgroups $L_2(16)$ in B.

To each of these groups we adjoin the unique involution commuting with the D_{30}. In each case we obtain a group which is larger than $L_2(16)$ (since it contains elements of order 34), but is still a proper subgroup of B (since the representation is reducible for this subgroup). It follows that no $L_2(16)$ is maximal in B. It is not too hard to tighten up this result to the following.

Proposition 5.1 *There are exactly two classes of* $5B$*-type* $L_2(16)$*s in* B*, each self-normalizing and contained in the maximal subgroup* $2^9.2^{16}.S_8(2)$*.*

It should also be noted that $L_2(31)$ can be generated in the same manner, and turns up four times in the list. In particular, we see that $L_2(31)$ contains representatives of both classes of $5B$-type A_5s, from which it immediately follows that it is self-normalizing.

Subgroups isomorphic to $L_2(25)$. The group $L_2(25)$ may be generated from A_5 by extending D_{10} to $5^2{:}2$. Indeed, $L_2(25)$ contains S_5, all of whose involutions are conjugate in $L_2(25)$, so such an A_5 must be contained in a copy of Th. Moreover, the 5^2-subgroup must be $5B$-pure, which implies from [8] that it is contained in the normal 5^{1+4} of the centralizer of any of its non-trivial elements.

In fact there are just four extensions of this type, none of which gives rise to $L_2(25)$. Since the Tits group ${}^2F_4(2)'$ contains $L_2(25)$, we obtain another proof that there is no subgroup ${}^2F_4(2)'$ containing $5B$-elements in B.

6 Subgroups isomorphic to $L_2(23)$

Any subgroup $L_2(23)$ can be generated by a Frobenius group 23:11 together with an involution inverting any particular element of order 11 in 23:11. We therefore proceed by first finding the 23-normalizer $2 \times 23{:}11$ inside the $2B$-centralizer $2^{1+22}{\cdot}Co_2$. Then we choose an element of order 11 in this group and find the group $D_{22} \times S_5$ of all elements which centralize or invert it. Finally, we run through the $1 + 10 + 15 = 26$ ways of extending 11 to D_{22} to see which extend 23:11 to $L_2(23)$. We obtain the following.

Proposition 6.1 *There is exactly one class of subgroups* $L_2(23)$ *in* B*. Any such subgroup is self-normalizing, and is contained in a maximal subgroup* Fi_{23}*.*

7 Subgroups isomorphic to $G_2(3)$

Our strategy for classifying subgroups isomorphic to $G_2(3)$ is to take representatives of the two conjugacy classes of subgroups $L_2(13)$, and in each case extend D_{14} to 7:6. Both types of $L_2(13)$ can be found inside the maximal subgroup $N(3A) \cong S_3 \times Fi_{22}{:}2$.

Now we need to find the centralizers of our two groups D_{14}, working inside the 7-normalizer $(7{:}3 \times 2{}^{\cdot}L_3(4).2).2$. The first of these centralizers turns out to be $2 \times A_5$, corresponding to an involution in class $2D$ in $L_3(4){:}2^2$. The other case, however, gives an involution in $L_3(4){:}2^2$-class $2B$, whose centralizer in $L_3(4)$ is $3^2{:}Q_8$. Since the pre-image of the latter group contains the 3-element which centralizes the $L_2(13)$, it follows that any extension of the second D_{14} to 7:6 necessarily centralizes a $3A$-element, and therefore any $G_2(3)$ containing the second $L_2(13)$ is contained in Fi_{22}.

Finally, then, we need to consider the 21 ways of extending the first D_{14} to 7:6. It turns out that only one of these extends $L_2(13)$ to $G_2(3)$. The others are easily eliminated by finding an element in the group whose order is not the order of any element of $G_2(3)$. Since this case centralizes S_3, we have proved the following result.

Proposition 7.1 *There are exactly two classes of subgroups $G_2(3)$ in B. The normalizers are $S_3 \times G_2(3)$ and $S_3 \times G_2(3){:}2$, both of which are contained in $S_3 \times Fi_{22}{:}2$.*

8 Subgroups isomorphic to $U_3(8)$

Our strategy here is to take a subgroup $3 \times L_2(8)$ and adjoin an involution inverting a diagonal element of order 9. Now the subgroups $L_2(8)$ in B were classified in [11], from which it is easy to see that there are just three classes of subgroups $3 \times L_2(8)$ in B, with normalizers as follows.

1. $L_2(8){:}3 \times 2^2 \times S_3$,
2. $L_2(8){:}3 \times S_3$,
3. $L_2(8) \times S_3$.

Any subgroup $3 \times L_2(8)$ is contained in $S_3 \times Fi_{22}{:}2$, and the second and third types can be found in the subgroup $S_3 \times 2^6 S_6(2)$ thereof.

Note that this implies that there are exactly three classes of $L_2(8)$ in $Fi_{22}{:}2$, becoming five classes in Fi_{22}. More precisely we have the following result.

Proposition 8.1 *The Fischer group Fi_{22} contains five classes of subgroups $L_2(8)$, with normalizers $L_2(8){:}3 \times 2$, $L_2(8){:}3$ (two classes), and $L_2(8)$ (two classes).*

In particular, there are two more classes than is claimed in [7]. At first sight, the classification of subgroups $G_2(3)$ and $S_6(2)$ in [7] depends on the classification of $L_2(8)$. However, closer inspection reveals that it really only depends on the classification of subgroups $L_2(8){:}3$, which were in fact listed correctly in [7]. Thus the lists of subgroups $G_2(3)$ and $S_6(2)$ of Fi_{22} given there are also (I believe) correct.

Returning now to the classification of subgroups $U_3(8)$ in the Baby Monster, we first show by explicit calculation that the second and third classes of $L_2(8)$ contain elements of Fi_{22}-class $9C$. (In fact, this can be obtained from the class fusion from $2^6{:}S_6(2)$ into Fi_{22}, which is available in the GAP library [6].) It then follows that the corresponding groups $3 \times L_2(8)$ contain elements of both B-classes $9B$ (inside $L_2(8)$) and $9A$ (the diagonal elements). But all cyclic subgroups of order 9 in $U_3(8)$ are conjugate, so these groups $3 \times L_2(8)$ cannot extend to $U_3(8)$ in the Baby Monster. We are therefore left with the case when we have an $L_2(8) \times 3$ of the first type.

This contains just $9B$-elements, and the subgroup generated by elements which invert a given $9B$-element has the shape $D_{18} \times 3^3{:}(2 \times S_4)$. Checking through the cases we easily eliminate all but two. Both groups are centralized by an involution, and are conjugate to each other, and to the known $U_3(8)$ in $2{\cdot}{}^2E_6(2){:}2$. Thus we have the following.

Proposition 8.2 *There is a unique conjugacy class of subgroups $U_3(8)$ in B. The normalizer of any such subgroup is $2 \times U_3(8){:}6 < 2{\cdot}{}^2E_6(2){:}2$.*

9 Subgroups isomorphic to $L_3(3)$

First note that the 3-central elements in $L_3(3)$ fuse to class $3B$ in B, since the centralizer is 3^{1+2}. Next, note that the normalizer of an element of order 13 in B is $S_3 \times 13{:}12 < S_3 \times Fi_{22}{:}2$. Now by looking in the chain of subgroups

$$13{:}3 < L_2(13) < G_2(3) < O_7(3) < Fi_{22}$$

we see that the elements of order 3 in this group 13:3 are of class $3D$ in $G_2(3)$, so class $3G$ in $O_7(3)$ and class $3D$ in Fi_{22}. Then from the character restriction given on p. 200 of [8] we see that both these elements and the diagonal elements of order 3 fuse to class $3B$ in B. We also have that there are two classes of subgroups 13:3 in B, with normalisers $S_3 \times 13{:}12$ and $3 \times 13{:}12$ respectively.

In particular, all 3-elements in any subgroup $L_3(3)$ are of class $3B$, and therefore the subgroups $3^2{:}2S_4$ of $L_3(3)$ are contained in one of the three classes of $3B^2$-normalizer in B. According to [8], these have the following shapes: $3^2.3^3.3^6.(S_4 \times 2S_4)$ (type (a)), $(3^2 \times 3^{1+4}).(2^2 \times 2A_4).2$ (type (b)), and $(3^2 \times 3^{1+4}).(2 \times 2S_4)$ (type (c)).

We divide the problem up into three cases, as follows. Case 1: the $L_3(3)$ contains a $3^2{:}2S_4$ of type (b). Case 2: the $L_3(3)$ contains a $3^2{:}2S_4$ of type (c). Case 3: both classes of $3^2{:}2S_4$ in $L_3(3)$ are of type (a).

We show first that case 1 cannot happen. Following the notation of [8], we take the 3^2 to be generated by $(0, 1, 1, 1)$ and $(0, i, i, j)$. Then its centralizer may be generated by $(1, 0, 0, 0)$, $(i, 0, 0, 0)$, $(0, 1, -1, 0)$, $(0, i, -i, 0)$ together with $\langle -, +, +, + \rangle$ and $\langle +, \bullet\!\!-\!\!-\!\!-\!\!\bullet, + \rangle$. This is then normalized by a group $2A_4$ generated by $\langle i, -k, -k, -i \rangle$ and $\langle \omega, \omega, \omega, \omega \rangle$, extending to $2S_4$ by adjoining

$$\frac{1}{2}\begin{pmatrix} 0 & i-j & j-i & 0 \\ j-i & 0 & 0 & i-j \\ i-j & 0 & 0 & i-j \\ 0 & j-i & j-i & 0 \end{pmatrix}.$$

In particular, we see by direct calculation (by hand!) that there is a unique class of subgroups $2S_4$ in here, in which the central involution is $\langle -, -, -, - \rangle$, which is in class $2D$ in B. On the other hand, all the outer involutions are conjugate to the one given above, which commutes with the element

$$\frac{1}{2}\begin{pmatrix} -1 & 1 & 1 & 1 \\ -1 & -1 & -1 & 1 \\ -1 & 1 & -1 & -1 \\ -1 & -1 & 1 & -1 \end{pmatrix}$$

of order 3, so cannot be in class $2D$. (If it were, its product with the central 3-element would be in class $6J$, which has centralizer $3^{1+4}.[2^{11}]$, and in particular centralizes no elements of order 3 outside the 3^{1+8}.) But this now contradicts the fact that all involutions in $L_3(3)$ are conjugate.

The other two cases are rather harder. Note first that in case 3, our hypothesis implies that two of the four elementary abelian 3^2-groups in the Sylow 3-subgroup 3^{1+2} of $L_3(3)$ are contained in the 3^{1+8}, and therefore the whole Sylow 3-group is in 3^{1+8}. In particular, all 3^2-subgroups are of type (a).

In both cases 2 and 3, we make heavy use of two maximal 3-local subgroups, namely $N(3B) \cong 3^{1+8}{:}2^{1+6}{\cdot}U_4(2){:}2$ and $N(3B^2) \cong 3^2.3^3.3^6{:}(S_4 \times 2{\cdot}S_4)$.

$L_3(3)$**s containing a** $3^2{:}2S_4$ **of type (c).** We show that there is a unique class of such subgroups $L_3(3)$, and each such subgroup is self-normalizing and maximal in B.

We may consider the 3^2 to be generated by $(0,1,1,1)$ and $(i,0,1,-1)$, centralized by $(i,0,0,0)$, $(0,1,0,0)$, $(1,0,-i,i)$ and $(0,i,i,i)$, together with $\langle +,-,-,- \rangle^*$, where * denotes the semilinear map $1 \mapsto 1, i \mapsto -i$. The normalizer is then generated by these elements together with $\langle \overset{i}{\bullet}\!\!\relbar\!\!\relbar\!\!\overset{i}{\bullet}, \overset{+}{\bullet}\!\!\relbar\!\!\relbar\!\!\overset{-}{\bullet} \rangle$, $\langle +, \bullet\!\!\relbar\!\!\relbar\!\!\bullet\!\!\relbar\!\!\longrightarrow \rangle$ and $\langle -,+, \bullet\!\!\relbar\!\!\relbar\!\!\bullet \rangle$.

Thus there are two classes of $3^2{:}2S_4$ of this type. One of these has outer involutions conjugate to $\langle -,+, \bullet\!\!\relbar\!\!\relbar\!\!\bullet \rangle$, which is in class $2C$ in B, so cannot be in any $L_3(3)$. Thus we must have one of the other class of $3^2{:}2S_4$, which contains a conjugate of $\langle -,-, \overset{-}{\bullet}\!\!\relbar\!\!\relbar\!\!\overset{-}{\bullet} \rangle^*$, which is in class $2D$ in B.

Now some straightforward hand calculations show that this $3^2{:}2S_4$ has trivial centralizer in B, as does its subgroup $3^2{:}(2 \times S_3) \cong 3^{1+2}{:}2^2$. The latter has two normal 3^2-subgroups, one of which is the one of type (c) whose normalizer we started with. The other turns out to be of type (a)—it may be possible to prove this by hand, but I used the computer to do this. In particular we have an embedding of our group $3^2{:}(2 \times S_3)$ in $3^2.3^3.3^6{:}(S_4 \times 2S_4)$, the normalizer of a 3^2-group of type (a), and we wish to find all groups $3^2{:}2S_4$ which lie between these two groups.

In fact, it turns out that there are just two such groups—it is not too hard to prove this by hand, but we also checked this on the computer. One of these contains $4E$-elements, whereas the $3^2{:}2S_4$ of type (c) does not. Since $L_3(3)$ contains a single class of elements of order 4, we can eliminate this case. Thus we have a unique group generated by two copies of $3^2{:}2S_4$, one of type (c), intersecting in $3^2{:}(2 \times S_3)$. We made this group, as described in detail below, and found that it was in fact isomorphic to $L_3(3)$. It is clear from the construction that it is self-normalizing. On the other hand, all subgroups $L_3(3)$ contained in previously known maximal subgroups have non-trivial centralizers. Hence this group is not contained in any previously known maximal subgroup, and is therefore maximal.

$L_3(3)$s in which all 3^2s are of type (a). We show that all such groups $L_3(3)$ have non-trivial centralizer.

In this case we adopt a different strategy. We start with a group 13:3, and extend a cyclic 3 subgroup to 3^2. As we have seen, there are two classes of subgroups 13:3 in B. Also, each cyclic 3 extends to just 2560 groups 3^2 of type (a). Thus there are 5120 groups $\langle 13{:}3, 3^2\rangle$ to consider. It turns out that most are easily shown not to be $L_3(3)$, and all of the few that remain are centralized by either 2 or S_3.

10 Subgroups $U_3(3)$

First we note from Norton's work on the Monster [5] that any $7A$-type $U_3(3)$ in M either has centralizer $(2^2 \times 3^2)2S_4$ or has type $(2B, 3B, 3B, \ldots)$. In the former case, the centralizer in B is a non-trivial soluble group, and therefore the normalizer of any such $U_3(3)$ in B is contained in a local subgroup. In the latter case, any such $U_3(3)$ contains an $L_2(7)$ of M-type $(2B, 3B, 7A)$. According to Norton, there is a single class of such $L_2(7)$ in M. The corresponding subgroups of B are given in [11]. There are two classes, and the $L_2(7)$-normalizers are $L_2(7){:}2 \times 2^2$ and $L_2(7){:}2 \times 2$ in the two cases. In both cases the involutions in the $L_2(7)$ are of class $2D$ in B.

The first type of $L_2(7)$. In $U_3(3)$ there is a maximal subgroup $L_2(7)$, which contains two classes of S_4, which fuse in $U_3(3)$. It follows that $U_3(3)$ can be generated by two (conjugate) subgroups $L_2(7)$, intersecting in S_4. Moreover, since in each case the normalizer of the $L_2(7)$ realises the outer automorphism, we may assume that the two copies of $L_2(7)$ are conjugate by an element of the centralizer of the S_4. Our strategy therefore is to find appropriate groups $H \cong L_2(7)$, and then find the centralizer in B of a subgroup $K \cong S_4$ of H, and investigate the groups $\langle H, H^x\rangle$ as x runs through a transversal for $C_B(H)$ in $C_B(K)$.

The centralizer of the S_4 turns out to have order 32, in which we find the subgroup of order 4 which centralizes the original $L_2(7)$. Therefore there are eight cases to consider. A small calculation shows that none of these groups is $U_3(3)$.

The second type of $L_2(7)$. Next we consider subgroups which contain an $L_2(7)$ of the second type, which has normalizer $2 \times L_2(7){:}2$ in B, contained in $C(2A)$.

We adopt a slightly different strategy here. Inside the $L_2(7)$ we take an S_3, and adjoin an element of class $3B$ which centralizes the S_3. It is not hard to see, by looking in the Monster, that there is a unique class of S_3 of type $(3B, 2D)$ in B, and that the centralizer of such an S_3 is a group of the shape $3^4{:}2^{1+4}D_{12}$. As a check, we note that this accounts for the full structure constant $\xi(2D, 2D, 3B) = 1/2^7.3^5$.

We actually found a subgroup of index 2 in the full centralizer of the S_3, which was enough for our purposes. A detailed analysis of this group then enables us to list the $3B$-elements it contains. There are 48 such elements (i.e. 24 such cyclic subgroups) in the normal 3^4. Outside this subgroup there are 288 cyclic subgroups to consider. We find that none of these cases gives rise to $U_3(3)$.

Proposition 10.1 *Every $U_3(3)$ in B has non-trivial centralizer.*

11 Subgroups $L_2(17)$

To classify subgroups isomorphic to $L_2(17)$, we take a group 17:8 and then extend the 8 to $8{:}2 \cong D_{16}$. First note that the normalizer of a cyclic group of order 17 in B is $(2^2 \times 17{:}8){\cdot}2 \cong \frac{1}{2}(D_8 \times 17{:}16)$. Thus there are three conjugacy classes of groups 17:8 in B. It turns out that one of these contains $8K$-elements while the other two contain $8M$-elements. To determine the number of ways of extending 8 to D_{16}, we use the structure constants of type $(2, 2, 8)$. The only relevant non-zero structure constants are $\xi(2C, 2C, 8K) = 512/98304$, $\xi(2D, 2D, 8K) = 3072/98304$, $\xi(2C, 2D, 8M) = \xi(2D, 2C, 8M) = 512/32768$ and $\xi(2D, 2D, 8M) = 2048/32768$. Of course, since all involutions in $L_2(17)$ are conjugate, the only dihedral groups we are interested in contain only involutions of class $2D$. However, it seemed to be more trouble than it was worth to use this fact in the calculations.

In fact, we verified computationally that the three classes of 17:8 contain elements of class $8K$, $8M$ and $8M$ respectively, but this can also be proved by character restriction, as follows. We have

$$(2^2 \times 17{:}8)^{\cdot}2 < (2^2 \times S_4(4){:}2)^{\cdot}2 < (2^2 \times F_4(2)){:}2,$$

in which the 8-element in 17:8 is of class $8AB$ in $S_4(4){:}2$. In $S_4(4){:}4$ these elements have square roots, which by character restriction are in class $16HI$ in $F_4(2){:}2$. Thus the elements of order 8 are in class $8K$ in $F_4(2)$. Now the ordinary 4371-character of B restricts to $2^2 \times F_4(2)$ as $833 + (1 + 1105) + (1 + 1105) + 1326$, where the brackets delimit the eigenspaces of the central 2^2. Thus, using character values and power maps, the $8K$-elements in $F_4(2)$ fuse to class $8K$ in B, while both diagonal classes of 8-elements fuse to $8M$ in B.

We split the calculation into three cases, according to which of the three elements of order 8 we are using.

The $8K$-case. Here the element of order 8 has normalizer of order $2^{17}.3$, and there are $2^{12}.3$ cosets which consist of inverting elements, and just 448 of these consist of involutions. We ran through all these 448 ways of extending the 8 to a group D_{16}. It turned out that there were just six groups $L_2(17)$ among these cases, two centralized by 2^2 and four with centralizer 2. The first two are therefore the two classes of $L_2(17)$ inside $F_4(2)$, which are interchanged by the outer automorphism, and have normalizers $2^2 \times L_2(17) < (2^2 \times F_4(2)){:}2$. Of the last four, two are centralized by each of the two $2A$-elements in the 17-centralizer. Now ${}^2E_6(2)$ contains four classes of subgroups $L_2(17)$, one of which is centralized by an S_3 of outer automorphisms, while the other three are permuted in the natural manner by this S_3. In the Baby Monster, a group 2 of outer automorphisms is realised, so there is the one class of $L_2(17)$ in $F_4(2){:}2$ that we have already seen, and one other, whose normalizer is $2 \times L_2(17)$, contained in $2^{\cdot 2}E_6(2){:}2$.

The first $8M$-case. Using the same method as before, we run through all the 384 inverting cosets which consist of involutions, and find four cases which give rise to $L_2(17)$. Each has trivial centralizer, which means they are all conjugate, and have normalizer $L_2(17){:}2$. We verified explicitly the existence of this subgroup $L_2(17){:}2$, by checking generators and relations in suitable elements of the group. This subgroup is not contained in any of the previously known maximal subgroups of B. It is therefore maximal.

The second $8M$-case. Using the same method as before, we then run through the 384 involutory cosets inverting the second $8M$-element, and find that none could extend 17:8 to $L_2(17)$. (There are however some cases which give $2 \times L_2(17)$, in which the $L_2(17)$ is of $8K$-type.)

To summarise the results of this section, we have

Proposition 11.1 *There are exactly three conjugacy classes of subgroups isomorphic to* $L_2(17)$ *in* B. *Two have type* $(8K, 9B)$ *and have normalizers* $2^2 \times L_2(17)$ *and* $2 \times L_2(17)$. *The third has type* $(8M, 9A)$ *and normalizer* $L_2(17){:}2$.

References

[1] J. H. Conway, R. T. Curtis, S. P. Norton, R. A. Parker and R. A. Wilson, *An ATLAS of Finite Groups*, Oxford University Press, 1985.

[2] B. Fischer, A note concerning {3,4}-transposition groups, *unpublished*.

[3] D. C. Hunt, The character table of the Baby Monster, *unpublished*.

[4] U. Meierfrankenfeld and S. V. Shpektorov, The maximal 2-local subgroups of the Monster and Baby Monster, *in preparation*.

[5] S. P. Norton, The anatomy of the Monster. I, in *The ATLAS 10 years on (ed. R. T. Curtis and R. A. Wilson)*, LMS Lecture Note Series, Cambridge University Press, 1997. (to appear)

[6] M. Schönert *et al.*, GAP—*Groups, Algorithms and Programming*, RWTH Aachen, 1995.

[7] R. A. Wilson, On maximal subgroups of the Fischer group Fi_{22}, *Math. Proc. Cambridge Philos. Soc.* **95** (1984), 197–222.

[8] R. A. Wilson, Some subgroups of the Baby Monster, *Invent. Math.* **89** (1987), 197–218.

[9] R. A. Wilson, A new construction of the Baby Monster, and its applications, *Bull. London Math. Soc.* **25** (1993), 431–437.

[10] R. A. Wilson, Some new subgroups of the Baby Monster, *Bull. London Math. Soc.* **25** (1993), 23–28.

[11] R. A. Wilson, The symmetric genus of the Baby Monster, *Quart. J. Math. (Oxford)* **44** (1993), 513–516.

[12] R. A. Wilson, More on maximal subgroups of the Baby Monster, *Arch. Math.* **61** (1993), 497–507.

[13] R. A. Wilson, The maximal subgroups of the Baby Monster, II, *in preparation.*