# CONJUGACY CLASSES IN SPORADIC SIMPLE GROUPS

Ibrahim A. I. Suleiman *

Department of Mathematics and Statistics,
Mu'tah University, 61710 Al-Karak, Jordan

Peter G. Walsh and Robert A. Wilson

School of Mathematics and Statistics,
The University of Birmingham,
Edgbaston, Birmingham B15 2TT, UK

**Abstract**

We obtain efficient programs for obtaining representatives of all the conjugacy classes of all the sporadic simple groups except the Monster and the Baby Monster.

## 1 Introduction

For many purposes, for example in calculating modular character tables (see [8], [10] etc.), or in finding particular subgroups of a group, it is useful to have representatives of the conjugacy classes of a given group readily to hand. While 'generic' simple groups tend to have a nicely parametrized set of conjugacy classes, this is not true for the 26 sporadic simple groups. In this paper we try to obtain a list of class representatives, starting from the 'standard generators' defined in [11], with (close to) as few multiplications of group elements as possible. For the groups of order less than $10^9$ we are able to produce a

*Currently on sabbatical at Department of Mathematics and Statistics, Sultan Qaboos University, P.O. Box 36, Al-Khod 123, Sultanate of Oman

complete list of conjugacy classes of elements, carefully distinguishing between automorphic classes, and between algebraically conjugate classes. For the rest of the groups, we do not distinguish between algebraically conjugate classes. For the largest Fischer group, $Fi'_{24}$, we also do not distinguish between automorphic classes. Finally, we have not considered the Baby Monster or the Monster. In all cases, the open problems here seem to be difficult.

Our notation follows the Atlas of Finite Groups [2]. The matrix calculations were performed using a version of Parker's Meat-axe [5] implemented by M. Ringe [7]. A few character calculations were carried out using GAP [9]. All the programs which we construct here are available on the world-wide-web under the URL http://www.mat.bham.ac.uk/atlas/ [12].

# 2 Identifying conjugacy classes

Notice that it is sufficient to produce a generator for a representative of each conjugacy class of maximal cyclic subgroups, as every conjugacy class is represented by a suitable power of such an element.

We first need to have enough invariants to be able to identify the conjugacy class to which a given element belongs. Usually this means having several representations of the group, and looking at orders and traces of elements and their powers. In practice, small permutation representations are the easiest to work with, and also (happily) tend to be very good at distinguishing classes, since elements in different conjugacy classes most often have different cycle types. However, we often need some other representations as well.

For example, for the groups $HN$, $Ly$, $Th$ and $J_4$ the smallest permutation representations are uncomfortably big, and we used the following matrix representations instead:

- $HN$: dimension 133 over $GF(5)$ and dimension 132 over $GF(4)$.
- $Ly$: dimension 111 over $GF(5)$.
- $Th$: dimension 248 over $GF(5)$.
- $J_4$: dimension 112 over $GF(2)$.

In most cases it is sufficient to calculate orders and traces of elements in these matrix representations, while occasionally it is necessary to calculate also the dimension of the fixed space of the element or a suitable power of it. In $J_4$ we first find a $2A$-element, for example as the 8th power of an element of order 16, and calculate its fixed space, which we find has dimension 62. Then when we find an involution whose fixed space has dimension 56, we know it must be in class $2B$. This solves all problems except distinguishing $12A$ from $12B$ (see Section 5 below). A similar method using the classes $5A$ and $5B$ in the Lyons group $Ly$ resolves all problems there, or alternatively, one can use the 651-dimensional representation over $GF(3)$.

In most other cases the cycle types of the elements in the smallest permutation representation were sufficient to distinguish the necessary classes of cyclic subgroups. (In the cases $M_{11}$, $J_1$, $M_{23}$ and $J_3$, only the orders of the elements are needed.) There are just six exceptions, where we used the following representations instead:

| | |
|---|---|
| $M_{22}$: | 77 points. |
| $HS$: | dimension 133 over $GF(5)$. |
| $Co_3$: | 276 points and dimension 22 over $GF(2)$. |
| $Fi_{22}$: | 3510 points and dimension 78 over $GF(5)$. |
| $Fi_{23}$: | dimension 253 over $GF(3)$. |
| $Co_1$: | dimension 24 for $2Co_1$ over $\mathbb{Z}$ and its skew square. |

Again, we occasionally needed to use the dimension of the fixed space to distinguish similar classes. For example, to distinguish $8C$ from $8A$ in $Co_3$ we used the rank of $1 + x$ in the 22-dimensional representation over $GF(2)$ as an invariant for the conjugacy class of the element $x$. A similar case was trying to distinguish $12E$ from $12F$ in $Fi_{22}$. Here we had to use the 176-dimensional representation of the double cover $2{}^{\cdot}Fi_{22}$ over $GF(3)$. However, each of the classes $12E$ and $12F$ lifts to two classes in the double cover, so the rank of $1 + x$ is no longer an invariant of the $Fi_{22}$ conjugacy class. Thus we use the (unordered) pair $\{\mathrm{rank}(1 + x), \mathrm{rank}(1 - x)\}$ as an invariant to distinguish the classes.

For compatibility with the Atlas of Brauer characters [3], we sometimes need to distinguish between algebraically conjugate classes, that is classes of elements which generate the same cyclic subgroups. We have done this systematically for all the groups of order less than $10^9$, which is as far as the Atlas of Brauer characters goes. For larger groups we do not usually have enough information to be able to resolve the ambiguities, and even when we do, there may be no definitive published labelling of the classes for us to use. Thus we did not consider it sensible to attempt to do so at this stage.

The method is simply to calculate the given elements in certain matrix representations, and calculate their traces. We then use [3] to determine the possibilities for the conjugacy classes. In essence this method is no different from that already used to distinguish conjugacy classes of elements of the same order, except that the questions tend to be more subtle and difficult to answer. Moreover, it is essential to use matrix representations, as such classes cannot be distinguished by permutation representations alone. Indeed, it is often necessary to use extremely large matrix representations, which is one reason why the problem in general is very hard.

In the tables below, in each case we have given one choice of labelling for the conjugacy classes, compatible with the Atlas of Brauer characters [3]. The image of our labelling under an outer automorphism is obviously also compatible with this Atlas, but there may be other such labellings as well.

# 3 Finding good words

Our first step is to determine the conjugacy classes of all the 'short' words in the standard generators. We then use a suitable subset of these as input to the next stage. If the generators $a$ and $b$ have orders 2 and 3, then the shortest essentially different words are described in [11]. More formally, we identify words in $a$ and $b$ with elements of the free product $C_2 * C_3$ in the obvious way, and say two words are essentially different if neither generates a cyclic subgroup containing a conjugate (in $C_2 * C_3$) of the other. We let $x = ab$, and $y = ab^2$. Then the first few essentially different short words are as follows:

$$x, xy, xxy, xxxy, xxyy, xxxxy, xxxyy, xxyxy, xxxxxy, xxxxyy,$$
$$xxxyxy, xxxyyy, xxyxyy, xxyyxy, xxxxxxy, xxxxxyy, xxxxyxy, xxxyxxy,$$
$$xxxxyyy, xxxyxyy, xxxyyxy, xxyxxyy, xxyxyxy, \text{ etc.}$$

If the generators have orders 2 and 4, then there are rather more essentially different short words (defined in the analogous way). We let $x = ab, y = ab^2$ and $z = ab^3$. Then the first few such words are as follows:

$$x, y, xy, xz, xxy, xxz, xyy, xyz, xzy, xzz, xxxy, xxxz, xxyy, xxyz,$$
$$xxzy, xxzz, xyxz, xyyz, xyzy, xzyy, \text{ etc.}$$

For generators of orders 2 and 5, we actually used the same list of words, while for larger orders, we did not make a minimal list, but just used all possible words.

At this stage we have produced a straight line program (i.e. a program without loops or jumps—in our case it consists simply of a sequence of multiplications) which makes a selection of elements in several classes of maximal cyclic subgroups. Moreover, as all the words for these elements are short, the program has a small number of steps. We next multiply together all pairs of elements produced so far, to see if any of these lie in new maximal cyclic subgroups. Usually there will be several, and we can add these new elements to the list, and again look at all products of pairs. Iterating this procedure very often produces a complete class list very quickly.

If not, then we can try products of three elements as well. Since there is a large number of such products, this will almost always be enough. In a handful of cases (such as the class $6E$ in $Fi_{23}$) we needed a product of four previous elements.

While it is not obvious that this procedure produces a program of the shortest possible length, inspection of the tables of results will show that at least it gets close. For example, in $Co_2$ the program consists of 24 multiplications, and it is easy to show that it is not possible to produce a program with fewer than 23 multiplications. In a few cases, such as $HS$, $McL$, $He$, it is clear that the minimum length has been attained.

## 4 An example

To see how this works in practice, we take the example of $J_2$. Here the standard generators have orders 2 and 3, with product of order 7 and commutator of order 12. Writing $a$ and $b$ for the generators, we therefore have $ab \in 7A$ and $abab^2 \in 12A$. Working with the permutation representations on 100 and 280 points to distinguish the classes, we find that $ab(abab^2)^3 \in 6B$ (since it has order 6 and is fixed-point-free on 100 points) and $ab(abab^2)^2ab^2 \in 10CD$ (since it has order 10 and is fixed-point-free on 280 points). To make these elements, we have to perform at least seven multiplications, so we might as well make all of the following elements:

$$\begin{aligned}
c = ab & & \\
d = cb &= ab^2 \\
e = cd &= abab^2 \\
f = ce &= ababab^2 \\
g = fe &= ab(abab^2)^2 \\
h = ge &= ab(abab^2)^3 \\
i = gd &= ab(abab^2)^2ab^2
\end{aligned}$$

Multiplying together all pairs of these elements, we find that $hi \in 10AB$. Multiplying this new element by all the previous ones does not produce any new conjugacy classes. We therefore replace $hi$ by $j = ih$, which is obviously in the same conjugacy class, and try again. This time we find that $k = bj \in 15A$, and then $l = bk \in 8A$, which completes the list of generators of maximal cyclic subgroups.

To sort out the algebraic conjugacies, it is sufficient to measure the traces on the elements of order 5 and 15 in the 6-dimensional and 14-dimensional representations over $GF(4)$.

In this example, our program has 10 lines, and it is easy to see that it cannot be done with fewer than 8 lines, since the calculation of $ab^2$ (or $bab$ or $b^2a$) is essential and does not produce a new conjugacy class. Indeed, once we have calculated $abab^2$ or an equivalent word, in class $12A$, all the new words which can be produced with one or two multiplications are equivalent to one of the above 'short words' of length at most 5 in $x$ and $y$. But these are easily checked to see that no new conjugacy class appears. Therefore our program is as short as possible.

## 5 Larger groups

The above methods work smoothly for the groups of order up to about $10^{16}$. When we try to extend these results to larger groups, we encounter a problem.

Specifically, we may not be able to make enough representations to distinguish classes of cyclic subgroups of the same order. The first case where this happens is in the Thompson group $Th$, where the two classes of cyclic subgroups of order 36, labelled $36A$ and $36BC$, have the same power maps and the same character values on all the representations that it is practical to make. However, these elements can be distinguished inside the involution centralizer, so having made an element $x$ of order 36, we next find the centralizer of its 18th power $x^{18}$, using for example Bray's algorithm [1]. This is a group of shape $2^{1+8}\cdot A_9$. Inside this group we find the centralizer of $x^9$, which is a group of shape $4.2^6.L_2(8){:}3$. Then $x$ is a $36A$-element if and only if it lies in the subgroup $4.2^6.L_2(8)$ of index 3. (We are grateful to John Bray for providing us with this improved method.) This therefore gives us a practical method of determining which class of cyclic subgroup a given element of order 36 is in. The precise method of determining whether $x$ is in the subgroup of index 3 will depend on the representation—for example, in the 248-dimensional representation over $GF(2)$, the full centralizer of $x^9$ has a 2-dimensional constituent, whose kernel is precisely this subgroup of index 3.

Two other difficult cases of the same type are the problems of distinguishing $12A$ from $12B$ in $J_4$, and $18D$ from $18G$ in $Fi_{23}$. In the former case, if $x$ is an element of class $12A$ or $12B$, then $x$ is in $12A$ if and only if $x^3$ lies in the normal $2^{1+12}$ subgroup of $C(x^6) \cong 2^{1+12}\cdot 3\cdot M_{22}{:}2$. In the latter case, if $x$ is an element of class $18D$ or $18G$, then $x$ is in $18D$ if and only if it lies in the normal subgroup $(2^2 \times 2^{1+8})U_4(2)$ of $C(x^9) \cong (2^2 \times 2^{1+8})(3 \times U_4(2)).2$. In both cases these questions can be resolved in a very similar way to the above.

It is worth asking whether we can do the same thing in the Baby Monster. Until very recently, the only representation available was the 4370-dimensional one over $GF(2)$, and there are many pairs of classes which cannot easily be distinguished in this representation. Now, however, we have the 4371-dimensional representations over $GF(3)$ and $GF(5)$ (see [6]), which means that character values can be determined modulo 30 by taking traces, which gives some hope that eventually we shall be able to complete this case also. Nevertheless, there are still some hard cases which we do not know how to solve. One of these is finding a $16F$-element and distinguishing it from class $16D$, where even the method described in this section seems not to work. Moreover, the amount of computing time required is several orders of magnitude greater than all of the previous cases put together.

Finally, perhaps, we should consider the Monster itself. We now have an effective computer construction (see [4]), in which it is possible to calculate the orders of (short) words in the generators. At present, however, we have no means of calculating traces (which in any case is not a good invariant, as the representation is over $GF(2)$), although it should be possible to devise a reasonable algorithm to do so. If this is combined with analogous constructions of the group over $GF(3)$ and/or $GF(5)$, then it may be possible to produce a partial list of class representatives even for the Monster. Another possibility may be to mimic the construction of [4], using $GF(7)$ instead of $GF(4)$, in order to calculate character values modulo 14. This would be sufficient to distinguish classes of cyclic subgroups of the same order, with the single exception of $27A$ and $27B$, which have the same character value and the same power maps.

# 6 Results

In the following tables, we give generators for representatives of the conjugacy classes of maximal cyclic subgroups. In all the groups of order less than $10^9$, we put a star after the letter denoting the conjugacy class of the given representative. The most efficient way of calculating the words is also given: simply make them in alphabetical order. In all cases, $a$ and $b$ denote the standard generators of the group, as defined in [11] or [12].

Machine-readable versions of these tables, in the form of small programs to make the given elements (or occasionally, obvious conjugates of them), are also available from [12].

Acknowledgements

We would like to thank the London Mathematical Society and the School of Mathematics and Statistics in the University of Birmingham, for support of a visit to Birmingham by the first author, during which this work was done. We also thank the SERC (now EPSRC) for a grant to purchase a computer on which most of the calculations were performed.

# References

[1] J. N. Bray, An improved algorithm for finding the centralizer of an involution, *to appear*.

[2] J. H. Conway, R. T. Curtis, S. P. Norton, R. A. Parker and R. A. Wilson, *An ATLAS of Finite Groups*, Clarendon Press, Oxford, 1985.

[3] C. Jansen, K. Lux, R. A. Parker and R. A. Wilson, *An Atlas of Brauer Characters*, Clarendon Press, Oxford, 1995.

[4] S. A. Linton, R. A. Parker, P. G. Walsh and R. A. Wilson, Computer construction of the Monster, *J. Group Theory* **1** (1998), 307–337.

Table 1: Classes in $M_{11}$

| Class | $5A$ | $6A$ | $8A^*B$ | $11A^*B$ |
|---|---|---|---|---|
| Words | $f = eb$ | $d = cb$ | $e = cd^2$ | $c = ab$ |

Table 2: Classes in $M_{12}$

| Class | $6A$ | $6B$ | $8A$ | $8B$ | $10A$ | $11A^*B$ |
|---|---|---|---|---|---|---|
| Words | $e = cd$ | $d = c^2b$ | $h = cedcb$ | $g = fd$ | $f = cecb$ | $c = ab$ |

[5] R. A. Parker, The computer calculation of modular characters (The 'Meat-axe'), *Computational Group Theory (ed. M. D. Atkinson)*, pp. 267–274. Academic Press, 1984.

[6] R. A. Parker and R. A. Wilson, Construction of Fischer's Baby Monster over fields of characteristic not 2, Preprint 99/13, School of Mathematics and Statistics, University of Birmingham.

[7] M. Ringe, *The C Meat-axe 2.3, documentation*, RWTH Aachen, 1995.

[8] F. Röhr, The 13-modular and 29-modular decomposition matrices for the Rudvalis sporadic simple group, Ph. D. thesis, RWTH Aachen, 1999.

[9] M. Schönert et al., *GAP 3.4 Manual (Groups, Algorithms, and Programming)*, RWTH Aachen, 1994.

[10] I. A. I. Suleiman, Modular representations of some finite simple groups, Ph. D. Thesis, University of Birmingham, 1990.

[11] R. A. Wilson, Standard generators for sporadic simple groups, *J. Algebra* **184** (1996), 505–515.

[12] R. A. Wilson *et al.*, A world-wide-web atlas of finite group representations, http://www.mat.bham.ac.uk/atlas.

Table 3: Classes in $J_1$

| Class | $6A$ | $7A$ | $10AB^*$ | $11A$ | $15AB^*$ | $19A^*BC$ |
|---|---|---|---|---|---|---|
| Words | $g = df$ | $c = ab$ | $f = de$ | $h = ef^2$ | $e = cdcb$ | $d = c^2b$ |

Table 4: Classes in $M_{22}$

| Class | $4B$ | $5A$ | $6A$ | $7A^*B$ | $8A$ | $11A^*B$ |
|---|---|---|---|---|---|---|
| Words | $h = c^2e$ | $d = cb$ | $e = cdb$ | $g = c^4dcd^2$ | $f = c^2d^2b$ | $c = ab$ |

Table 5: Classes in $J_2$

| Class | $6B$ | $7A$ | $8A$ | $10AB^*$ | $10C^*D$ | $12A$ | $15A^*B$ |
|---|---|---|---|---|---|---|---|
| Words | $e = cd^3$ | $c = ab$ | $i = bh$ | $g = fe$ | $f = cd^2cb$ | $d = c^2b$ | $h = bg$ |

Table 6: Classes in $M_{23}$

| Class | $6A$ | $8A$ | $11AB^*$ | $14AB^*$ | $15A^*B$ | $23AB^*$ |
|---|---|---|---|---|---|---|
| Words | $d = cb$ | $e = cd$ | $g = cf$ | $h = bgd$ | $f = ce$ | $c = ab$ |

Table 7: Classes in $HS$

| Class | $5C$ | $6A$ | $7A$ | $8A$ | $8B$ | $8C$ |
|---|---|---|---|---|---|---|
| Words | $h = fg$ | $l = kh$ | $g = ce$ | $f = eb$ | $j = ci$ | $m = el$ |
| Class | $10B$ | $11A^*B$ | $12A$ | $15A$ | $20A^*B$ | |
| Words | $d = cb$ | $c = ab$ | $k = hj$ | $e = cd$ | $i = ah$ | |

Table 8: Classes in $J_3$

| Class | $8A$ | $9AB^*C$ | $10A^*B$ | $12A$ | $15AB^*$ | $17AB^*$ | $19A^*B$ |
|---|---|---|---|---|---|---|---|
| Words | $f = de$ | $d = c^2b$ | $h = gcb$ | $i = he$ | $g = cbf$ | $e = cd$ | $c = ab$ |

Table 9: Classes in $M_{24}$

| Class | $8A$ | $10A$ | $11A$ | $12A$ | $12B$ |
|---|---|---|---|---|---|
| Words | $f = ce$ | $k = ed$ | $h = gcb$ | $e = cd$ | $d = c^2b$ |
| Class | $14AB^*$ | $15A^*B$ | $21A^*B$ | $23A^*B$ | |
| Words | $g = fcb$ | $i = cf$ | $j = ci$ | $c = ab$ | |

Table 10: Classes in $McL$

| Class | $5B$ | $6B$ | $8A$ | $9AB^*$ | $11A^*B$ | $12A$ | $14A^*B$ | $30A^*B$ |
|---|---|---|---|---|---|---|---|---|
| Words | $j=gi$ | $i=hd$ | $g=ebd$ | $f=ce$ | $c=ab$ | $d=cb$ | $e=cd$ | $h=gb$ |

Table 11: Classes in $He$

| Class | $8A$ | $10A$ | $12A$ | $12B$ | $14CD$ |
|---|---|---|---|---|---|
| Words | $f=db$ | $h=gd$ | $j=bi$ | $d=cb$ | $k=hd$ |
| Class | $15A$ | $17AB$ | $21AB$ | $21CD$ | $28AB$ |
| Words | $e=cd$ | $c=ab$ | $l=jf$ | $i=gf$ | $g=ef$ |

Table 12: Classes in $Ru$

| Class | $8C$ | $10B$ | $12B$ | $14ABC$ | $15A$ | $16AB$ |
|---|---|---|---|---|---|---|
| Words | $m=kci$ | $l=cjbk$ | $h=abg$ | $c=ab^2$ | $g=d^2c$ | $k=bj$ |
| Class | $20A$ | $20BC$ | $24AB$ | $26ABC$ | $29AB$ | |
| Words | $f=ae$ | $j=idc$ | $i=ch$ | $e=bdc$ | $d=abc$ | |

Table 13: Classes in $Suz$

| Class | $8B$ | $8C$ | $10B$ | $11A$ | $12B$ | $12C$ | $12D$ | $12E$ |
|---|---|---|---|---|---|---|---|---|
| Words | $h=gcb$ | $g=ed$ | $j=ci$ | $q=jl$ | $e=cd$ | $k=ej$ | $l=di$ | $m=eh$ |
| Class | $13AB$ | $14A$ | $15AB$ | $15C$ | $18AB$ | $20A$ | $21AB$ | $24A$ |
| Words | $c=ab$ | $i=ch$ | $d=c^2b$ | $n=cj$ | $o=fg$ | $p=h^2ec^2$ | $f=ce$ | $r=no$ |

Table 14: Classes in $O'N$

| Class | $7B$ | $11A$ | $12A$ | $15AB$ | $16AB$ |
|---|---|---|---|---|---|
| Words | $k=fj$ | $c=ab$ | $f=eb$ | $l=kh$ | $i=ceh$ |
| Class | $16CD$ | $19ABC$ | $20AB$ | $28AB$ | $31AB$ |
| Words | $j=di$ | $d=cb$ | $e=cd$ | $h=fce$ | $g=ced$ |

Table 15: Classes in $Co_3$

| Class | $6D$ | $6E$ | $8C$ | $9B$ | $10B$ | $12A$ |
|---|---|---|---|---|---|---|
| Words | $m = a^2bh$ | $h = df$ | $s = jm$ | $q = oa^2b^2$ | $p = ck$ | $r = jq$ |
| Class | $12C$ | $14A$ | $15B$ | $18A$ | $20AB$ | $21A$ |
| Words | $n = gi$ | $c = ab$ | $e = (ac)^2b$ | $j = bg$ | $o = bk$ | $f = de$ |
| Class | $22AB$ | $23AB$ | $24A$ | $24B$ | $30A$ | |
| Words | $k = dg$ | $g = acf$ | $d = cb = ab^2$ | $i = a^2b^2f$ | $l = eg$ | |

Table 16: Classes in $Co_2$

| Class | $6F$ | $8E$ | $8F$ | $11A$ | $12A$ | $12D$ | $12E$ | $12F$ |
|---|---|---|---|---|---|---|---|---|
| Words | $t = kq$ | $r = hl$ | $w = os$ | $i = de$ | $v = br$ | $x = g^2o$ | $n = ef$ | $m = bd$ |
| Class | $12G$ | $12H$ | $14BC$ | $16A$ | $16B$ | $18A$ | $20A$ | $20B$ |
| Words | $o = eh$ | $e = dcb$ | $p = hi$ | $q = fk$ | $u = dr$ | $h = cf$ | $k = cbe$ | $l = gh$ |
| Class | $23AB$ | $24A$ | $24B$ | $28A$ | $30A$ | $30BC$ | | |
| Words | $f = eb$ | $d = c^2b$ | $s = dl$ | $c = ab$ | $g = cd$ | $j = df$ | | |

Table 17: Classes in $Fi_{22}$

| Class | $6E$ | $6F$ | $6J$ | $8D$ | $9C$ | $12A$ | $12C$ |
|---|---|---|---|---|---|---|---|
| Words | $aen^2t$ | $agps$ | $fso$ | $ck$ | $d = cb$ | $brfs$ | $u = enm$ |
| Class | $12D$ | $12E$ | $12F$ | $12G$ | $12H$ | $12I$ | $12J$ |
| Words | $t = eoi$ | $emi$ | $kjg$ | $p = in$ | $r = ip$ | $s = rab$ | $q = pd$ |
| Class | $12K$ | $13AB$ | $14A$ | $16AB$ | $18AB$ | $18C$ | $18D$ |
| Words | $o = ke$ | $b$ | $j = bg$ | $i = bf$ | $h = abf$ | $m = dg$ | $l = cg$ |
| Class | $20A$ | $21A$ | $22AB$ | $24A$ | $24B$ | $30A$ | |
| Words | $g = fd$ | $c = ab^2$ | $k = fg$ | $n = kl$ | $f = abe$ | $e = abd$ | |