Construction of exceptional covers of generic groups

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Abstract

We describe the construction of explicit representations of some of the exceptional covers of generic groups, that is, covers whose existence is not explained by the general theory of such groups.

1 Introduction

A number of groups of Lie type have so-called 'exceptional covers', that is, there is a non-trivial p-part to their Schur multiplier, where p is the defining characteristic of the group. If we neglect those cases which can be explained by exceptional isomorphisms (for example, the double cover of $L_3(2)$ is explained by the isomorphism $L_3(2) \cong L_2(7)$), we obtain the following list of cases:

$$4^{2\cdot}L_3(4), 2^{\cdot}U_4(2), 3^{2\cdot}U_4(3), 2^{2\cdot}U_6(2), 2^{2\cdot}Sz(8), 2^{\cdot}S_6(2), 3^{\cdot}O_7(3),$$

$$2^{2\cdot}O_8^+(2), 3^{\cdot}G_2(3), 2^{\cdot}G_2(4), 2^{\cdot}F_4(2) \text{ and } 2^{2\cdot 2}E_6(2),$$

in addition to the exceptional covers $3{}^{\cdot}A_6$ and $3{}^{\cdot}A_7$ of alternating groups.

Many of these can be obtained as subgroups of sporadic groups, such as

$$\begin{array}{ccccccc} & & & & 2{}^{\cdot}G_2(4) & < & 2{}^{\cdot}Suz \\ & & 3{}^{\cdot}G_2(3) & < & 3{}^{\cdot}O_7(3) & < & 3{}^{\cdot}Fi_{22} \\ & & 2{}^{\cdot}U_4(2) & < & 2{}^{\cdot}S_6(2) & < & Co_3 \\ 3{}^{\cdot}A_6 & < & 3{}^{\cdot}A_7 & < & 3^2{}^{\cdot}U_4(3) & < & 3{}^{\cdot}Suz \\ & & & & 2^2{}^{\cdot}U_6(2) & < & 2{}^{\cdot}Fi_{22} \end{array}$$

Others need to be constructed from scratch in the same way that we have constructed many representations of the sporadic groups (see, for example, [10], [11]). In addition, many of these groups have also non-trivial 'generic' covers, giving rise to the following additional groups:

$$(3 \times 4^2){}^{\cdot}L_3(4), (4 \times 3^2){}^{\cdot}U_4(3), (3 \times 2^2){}^{\cdot}U_6(2), \text{ and } 6{}^{\cdot}O_7(3).$$

We also exploit the idea of 'standard generators' [15] in two ways. First, given a pair (a, b) of standard generators for a group G (usually simple), we can find images (a', b') of (a, b) under outer automorphisms, and hence construct representations of $\mathrm{Aut}(G)$. Second, such an automorphism of G may be lifted to a 'near-automorphism' of a covering group; if there is a double cover $2{}^{\cdot}G$, for example, this may sometimes be used to construct a cover $2^2{}^{\cdot}G$, if one exists.

Our notation follows the Atlas of Finite Groups [3]. The matrix calculations were performed using a version of Parker's Meat-axe [5] implemented by M. Ringe [7]. A few character calculations were carried out using GAP [8]. All the representations which we construct here are available on the world-wide-web under the URL http://www.mat.bham.ac.uk/atlas/ [16]. They are listed in Table 1, though a more up-to-date list can be found by consulting the above-mentioned web-page.

2 $2^2{}^{\cdot}Sz(8)$

We note first that a double cover $2{}^{\cdot}Sz(8)$ is contained in the orthogonal group $2{}^{\cdot}O_8^+(5)$, and explicit generators are given in the ATLAS [3]. In matrix form the three generators A, B and C are as in Table 2.

In order to construct the 'other' double covers, it is useful to define 'standard generators' for the group $Sz(8)$. We let (a, b) be elements of $Sz(8)$

Table 1: The constructed representations

We list the degree d and the underlying field $GF(q)$ of the representations of the group G.

G	d	q
$2^2{\cdot}Sz(8){:}3$	24	5
$2^2{\cdot}U_6(2){:}S_3$	168	3
$2^2{\cdot}O_8^+(2){:}S_3$	24	3
$12_1{\cdot}L_3(4)$	24	25 & 49
$12_2{\cdot}L_3(4)$	12	49
$12_1{\cdot}U_4(3)$	84 & 132	25
$12_2{\cdot}U_4(3)$	36	25
$2{\cdot}F_4(2){:}2$	52	25
$2^{2\cdot 2}E_6(2){:}S_3$	1706	2

such that a has order 2, b has order 4, ab has order 5, $ababb$ has order 7 and $ababababbababbabb$ has order 13. Then it is not difficult to show that the pair (a, b) is unique up to automorphisms, and thus there are exactly three conjugacy classes of such pairs in $Sz(8)$.

Computing first in the simple group $Sz(8)$, we can find words u and v in a and b such that $a' = u(a, b)$ and $b' = v(a, b)$ give a pair (a', b') automorphic to, but not conjugate to, the pair (a, b). Suitable words are $u(a, b) = (ab)^{-4}a(ab)^4$ and $v(a, b) = (abb)^{-4}b(abb)^4$. Then clearly $a'' = u(a', b')$ and $b'' = v(a', b')$ give a pair (a'', b'') in the third conjugacy class. Lifting to the double cover $2{\cdot}Sz(8)$, represented as 8×8 matrices, denote suitable pre-images of these elements by the corresponding Greek letters.

In terms of the Atlas generators A, B, C we may take $\alpha = B^C$, $\beta = A$. Now from an alternative viewpoint we can consider the three pairs of matrices (α, β), (α', β') and (α'', β'') as being pre-images of the same pair (a, b) of elements of $Sz(8)$, and thus we obtain representations of the three double covers of $Sz(8)$. It is then clear that the 16×16 matrices $\begin{pmatrix} \alpha & 0 \\ 0 & \alpha' \end{pmatrix}$ and $\begin{pmatrix} \beta & 0 \\ 0 & \beta' \end{pmatrix}$ generate $2^2{\cdot}Sz(8)$.

Table 2: Generators for $2{\cdot}Sz(8)$

$$A = \begin{pmatrix} 0&0&0&0&0&1&0&0 \\ 0&0&0&0&0&0&4&0 \\ 0&0&0&1&0&0&0&0 \\ 0&0&1&0&0&0&0&0 \\ 0&0&0&0&0&0&0&4 \\ 1&0&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0&0 \\ 0&0&0&0&1&0&0&0 \end{pmatrix}, B = \begin{pmatrix} 1&0&0&0&0&0&0&0 \\ 0&0&1&0&0&0&0&0 \\ 0&0&0&1&0&0&0&0 \\ 0&0&0&0&1&0&0&0 \\ 0&0&0&0&0&1&0&0 \\ 0&0&0&0&0&0&1&0 \\ 0&0&0&0&0&0&0&1 \\ 0&1&0&0&0&0&0&0 \end{pmatrix},$$

$$C = \begin{pmatrix} 2&1&1&1&1&1&1&1 \\ 1&4&3&3&4&1&0&3 \\ 1&3&3&4&1&0&3&4 \\ 1&3&4&1&0&3&4&3 \\ 1&4&1&0&3&4&3&3 \\ 1&1&0&3&4&3&3&4 \\ 1&0&3&4&3&3&4&1 \\ 1&3&4&3&3&4&1&0 \end{pmatrix}$$

By suitable choice of pre-images we could arrange that the 24×24 matrices

$$\mathbf{A} = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \alpha' & 0 \\ 0 & 0 & \alpha'' \end{pmatrix} \text{ and } \mathbf{B} = \begin{pmatrix} \beta & 0 & 0 \\ 0 & \beta' & 0 \\ 0 & 0 & \beta'' \end{pmatrix}$$

also generate $2^2{\cdot}Sz(8)$ rather than $2 \times 2^2{\cdot}Sz(8)$, although this is not really important at this stage, as we can adjust by the scalar -1 at a later stage in the calculations.

Now we have

$$u(\mathbf{A}, \mathbf{B}) = \begin{pmatrix} \alpha' & 0 & 0 \\ 0 & \alpha'' & 0 \\ 0 & 0 & \alpha''' \end{pmatrix}$$

and

$$v(\mathbf{A}, \mathbf{B}) = \begin{pmatrix} \beta' & 0 & 0 \\ 0 & \beta'' & 0 \\ 0 & 0 & \beta''' \end{pmatrix},$$

where (α''', β''') is conjugate to (α, β). Thus the pair $(u(\mathbf{A}, \mathbf{B}), v(\mathbf{A}, \mathbf{B}))$ is conjugate to $(\mathbf{A}, \mathbf{B})$ in the general linear group. The usual standard basis calculation (see [5]) can then be used to find a matrix $\mathbf{M}$ which realises this conjugation. This action clearly cycles the three double covers of $Sz(8)$, and therefore realises the outer automorphism of $2^2{\cdot}Sz(8)$. Adjusting by a scalar if necessary, we obtain a matrix group isomorphic to $2^2{\cdot}Sz(8){:}3$.

3 Covers of $U_6(2)$ and $O_8^+(2)$

A similar procedure can be used to construct $2^2{\cdot}U_6(2){:}3$, given a faithful representation of $2{\cdot}U_6(2)$. A suitable representation is the 56-dimensional one over $GF(3)$, which can be obtained from the 77-dimensional representation of $Fi_{22}{:}2$ found in the library [16]. We defined standard generators of $U_6(2)$ to be (a, b) where $a \in 2A$, b has order 7, ab has order 11, and abb has order 18, and used the outer automorphism of order 3 defined by $a' = (abb)^{-4}a(abb)^4$ and $b' = ((ab)^3bab)^{-1}b(ab)^3bab$.

To extend to $2^2{\cdot}U_6(2){:}S_3$ we apply a similar argument, using two classes of 'standard generators' of $U_6(2){:}3$, interchanged by the outer automorphism of order 2. In this case, however, the 168-dimensional representation of $2^2{\cdot}U_6(2){:}3$ extends to $2^2{\cdot}U_6(2){:}S_3$, so the calculation is actually rather easier. If we denote the two pairs of generating matrices by (γ, δ) and (γ', δ'), then all we have to do is find a matrix conjugating (γ, δ) to (γ', δ'), and adjust by a scalar if necessary.

Full details of this case are described in [12], including a construction of the full 'envelope' $(3 \times 2^2){}^{\cdot}U_6(2)$ in characteristic 7.

The case of $O_8^+(2)$ is virtually identical. We start with $2{}^{\cdot}O_8^+(2) = W(E_8)'$, a representation of which is easy to write down from the ATLAS [3], and apply the same methods, to construct first $2^2{}^{\cdot}O_8^+(2){:}3$ and then $2^2{}^{\cdot}O_8^+(2){:}S_3$. We have carried out this construction over the field of order 3, but it equally well can be done over any field of characteristic not 2.

4 Covers and automorphisms of $L_3(4)$ and $U_4(3)$

Similar methods can be used in the case of $L_3(4)$, but there are extra complications due to the size of the multiplier and automorphism group. In particular, the fields required to write the representations are often large, and there are big problems with isoclinism. Specifically, it is often very difficult to get rid of unwanted scalars.

To begin with, we define standard generators of $L_3(4)$ to be a and b with a, b, ab, abb of orders 2, 4, 7 and 5 respectively.

We write down generators for $3{}^{\cdot}L_3(4) \cong SL_3(4)$ as 3×3 matrices over $GF(4)$. One of the fourfold covers $4_1{}^{\cdot}L_3(4)$ can be obtained as a subgroup of $4{}^{\cdot}M_{22}$, a representation of which was constructed in [11].

To obtain the other fourfold cover, we first find words giving images under the various automorphisms of the standard generators. As with the group $Sz(8)$ discussed above, this can be interpreted as giving another cover ${4_1}'{}^{\cdot}L_3(4)$. By tensoring together representations of these different covers, we obtain representations of $4_2{}^{\cdot}L_3(4)$.

By changing characteristic via permutation representations when necessary, we can tensor together representations of the triple and fourfold covers to obtain the 12-fold covers, and so on.

Again, for the group $U_4(3)$ we encounter problems of the same order, which we have not completely solved. By defining standard generators for $U_4(3)$ as a of order 2 and b in class $6A$, with ab of order 7 and $ab(ababb)^2$ of order 5, we can combine together representations from various places. The exceptional cover $3^2{}^{\cdot}U_4(3)$ can be obtained as a subgroup of Conway's group via $3{}^{\cdot}Suz$. The generic cover $4{}^{\cdot}U_4(3)$ arises in the natural representation of the unitary group $SU_4(3) \cong 4{}^{\cdot}U_4(3)$. We now need to obtain permutation

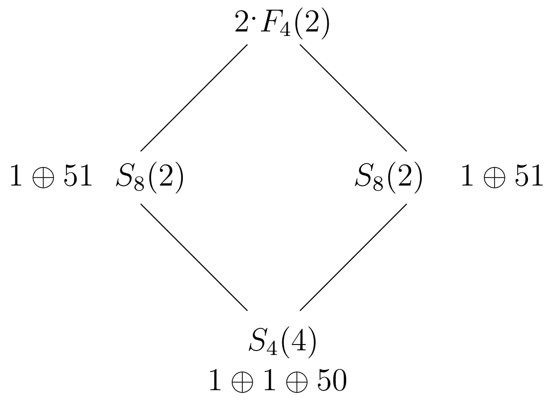

Figure 1: Generating $2{\cdot}F_4(2)$ from $S_8(2)$

representations for these groups in order to change the characteristic of the representation, and eventually construct the full cover $(3^2 \times 4){\cdot}U_4(3)$.

5 The double cover of $F_4(2)$

So far, we have managed to obtain all our representations directly from existing representations of related groups, with no construction more complicated than adjoining an outer automorphism. This time, however, we have to construct a representation 'from scratch'. Our plan is, following the method described in [6], to amalgamate two subgroups H and K over their intersection L. As the method is by now fairly standard, we do not give full details.

By using information in [4] about modular characters of various groups, it seemed most sensible to try to construct the 52-dimensional representation of $2{\cdot}F_4(2)$ in characteristic 5. This allows us to construct the group from two subgroups $S_8(2)$ intersecting in $S_4(4)$, in such a way that all the representations encountered are completely reducible, as shown in Figure 1. A very similar approach may be used to generate a group isoclinic to $2{\cdot}F_4(2){:}2$, by adjoining an element conjugating one copy of $S_8(2)$ to the other. Indeed, the latter approach is the one we actually followed.

The 51-dimensional representation of $S_8(2)$ can be obtained as follows. Note first that the permutation representation on 2295 points has ordinary character $1a + 51a + 135a + 918a + 1190a$ (see [9, p. 157]). Thus the desired representation may be obtained as a constituent of the reduction modulo 5 of this permutation representation, either by chopping directly with the Meat-axe [5], or with the aid of the condensation method (see [9] or [13]). The permutation representation itself is the representation on the maximal isotropic subspaces of the natural module, and can therefore be readily obtained. In practical terms it was easiest to find it by permuting a suitable orbit of vectors in the exterior fourth power of the natural representation.

We carry out a random search in $S_8(2)$ for a pair of standard generators (c, d) of a subgroup $S_4(4)$. We then find words in c and d which give images c' and d' under an outer automorphism of order 4.

We next use the standard basis method to find a matrix m conjugating (c, d) to (c', d'). Finally we run through all matrices mx where x is in the centralizer of (c, d), to see if $\langle mx, c, d\rangle$ could be of the shape $2{}^{\cdot}(F_4(2) \times 2){}^{\cdot}2$. We eliminate all cases except one, and therefore this last case must generate a group of the required shape $2{}^{\cdot}(F_4(2) \times 2){}^{\cdot}2$.

Extending the field to $GF(25)$, we can multiply our new element mx by a suitable scalar, to give an element e, say, so that $\langle c, d, e\rangle$ is actually $2{}^{\cdot}F_4(2){:}2$. As usual, we prefer to have a pair of 'standard' generators for the group, rather than three rather ill-defined generators. We work first modulo the centre.

Using GAP [8] to calculate structure constants in the character table of $F_4(2){:}2$, we found that there are just four conjugacy classes of pairs of elements (f, g) with $f \in 2E$, $g \in 3AB$ and fg of order 40. By a random search we found pre-images in $2{}^{\cdot}F_4(2){:}2$ of representatives of all four classes, and found that exactly one of these pairs generated the whole group. This can be distinguished from the others by the fact that in this case alone, $fgfgfgg$ has order 10 in $F_4(2){:}2$. We take these to be our standard generators for the group.

Lifting now to $2{}^{\cdot}F_4(2){:}2$, there are four possible pre-images $(\tilde{f}, \tilde{g})$, but these are fused in pairs by the outer automorphism which negates all elements in $2{}^{\cdot}F_4(2){:}2 \setminus 2{}^{\cdot}F_4(2)$. Thus there are just two types of pairs $(\tilde{f}, \tilde{g})$, and we choose the one in which $\tilde{g}$ has order 3 to be our standard generators for $2{}^{\cdot}F_4(2){:}2$.

Similarly we find the subgroup $2{}^{\cdot}F_4(2)$. We define standard generators of $F_4(2)$ to be (x, y) where $x \in 2C$, $y \in 3C$, xy has order 17, and $(xy)^4yxyxyyxyy$ has order 13. It is easy to find pre-images in $2{}^{\cdot}F_4(2)$ of such a pair of standard generators. We take $(\tilde{x}, \tilde{y})$ where $\tilde{y}$ has order 3 and $\tilde{x}\tilde{y}$ has order 17.

John Bray has used a slightly more sophisticated version of this method to construct the 52-dimensional representation of this group in characteristic zero (see [1]).

6 Covers of ${}^2E_6(2)$

The triple cover $3{}^{\cdot}{}^2E_6(2)$ can be obtained from the general theory of groups of Lie type. In fact, we took $3{}^{\cdot}E_6(4)$ as a subgroup of $E_8(4)$, generators of which were obtained from Carter's book [2]. Then a random search among groups generated by suitable elements of orders 2 and 3 eventually produced a subgroup $3{}^{\cdot}{}^2E_6(2)$.

The double cover is more difficult to obtain. A representation in characteristic 2 is available, as this group is a subgroup of the Baby Monster, a representation of which was constructed in [14]. The smallest subquotient on which the group acts faithfully has dimension 1704. On the other hand, it would also be interesting to construct a faithful irreducible representation over a field of odd characteristic.

First, however, we consider the $GF(2)$ representation. We define standard generators for ${}^2E_6(2)$ to be $a \in 2B$ and $b \in 3C$, with ab of order 19, and $(ab)^3b$ of order 33. It is then possible to show that the pair (a', b'), defined by $a' = (ab)^{-2}b(ab)^2$ and $b' = (abb)^{-6}b(abb)^6$, is the image of (a, b) under an outer automorphism of order 3. In just the same way as with $U_6(2)$, therefore, we can construct a representation of $2^{2\cdot 2}E_6(2){:}3$ in $3 \times 1704 = 5112$ dimensions over $GF(2)$. If we adjoin also the outer automorphism of order 2 which maps (a, b) to $(a, (abb)^{-9}b(abb)^9)$, then we obtain the full group $2^{2\cdot 2}E_6(2){:}S_3$.

In fact, however, we can do much better than this. The 1704-dimensional module is uniserial, with constituents of dimensions 1, 1702 and 1 in that order. If we write this with respect to a basis which exhibits this structure, and such that the irreducible constituents are themselves written with respect to a standard basis, then the 'glue' between the constituents is given by a column vector and a row vector, of length 1702, in each generating matrix. If we now apply the outer automorphism, and again write our representation with respect to a standard basis defined by the new generators a', b', then we find that the constituents remain the same, but the 'glue' has changed. By cutting and pasting the appropriate row and column vectors, we can construct a 1706-dimensional representation of the full cover $2^{2\cdot 2}E_6(2)$. This can now be extended to $2^{2\cdot 2}E_6(2){:}S_3$ in the usual way.

In the odd characteristic case, we need to make a representation from scratch. Our tentative plan is to amalgamate two subgroups $2{}^{\cdot}F_4(2)$ intersecting in $S_8(2)$. The representation we attempt to construct is that of dimension 2432 over $GF(3)$. This restricts to $2{}^{\cdot}F_4(2)$ as $52+2380$, and both of these constituents occur in the permutation representation of degree 139776, on the cosets of $S_8(2)$. We first made this permutation representation, by permuting a suitable orbit of vectors in the 52-dimensional representation over $GF(5)$. We condensed this representation modulo 3 over a subgroup D_{56}, and then we 'uncondensed' the two required modules.

The next step is to restrict to the subgroup $S_8(2)$. There are now two problems. One is that there are two classes of subgroups $S_8(2)$, which act differently on the module, and only one of these is in a second copy of $2{}^{\cdot}F_4(2)$. And even when we have the right one, the structure of the module for this subgroup is rather complicated. The module structure for the subgroup which appears to be the right one is as follows.

$$\begin{array}{c} 1 \\ 50 \\ 1 \end{array} \oplus 1 \oplus \begin{array}{c} 50 \\ 1+1225 \\ 50 \end{array} \oplus \begin{array}{c} 135 \\ 783 \\ 135 \end{array}.$$

It may be possible to complete this construction of the 2432-dimensional representation in characteristic 3, but for practical calculations it will be very much slower than the characteristic 2 representation constructed above. Moreover, to extend to $2^{2\cdot 2}E_6(2){:}S_3$ would require a representation of degree $3 \times 2432 = 7296$ which is really too large to be useful at this stage.

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