The symmetric genus of the Fischer group Fi_{23}

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Abstract

We show that the sporadic simple group Fi_{23} is generated by an element of order 2 and an element of order 3, whose product has order 8. Since Fi_{23} is not a Hurwitz group, we can deduce the symmetric genus of the group.

In [4] the symmetric genus of a finite group G is defined to be the smallest integer g such that G acts faithfully on a closed orientable surface of genus g. The Riemann–Hurwitz formula implies that this minimum is attained by minimising

$$\sum_{i=1}^{n}\left(1-\frac{1}{a_i}\right)$$

where G is generated by elements x_i satisfying $\prod_{j=1}^{n} x_j = 1$ and ${x_i}^{a_i} = 1$ for all i. In most, but not all, cases, this minimum is attained for $n = 3$, and we have

$$g = 1 + \frac{1}{2}|G|(1 - \frac{1}{r} - \frac{1}{s} - \frac{1}{t}),$$

where

$$\frac{1}{r} + \frac{1}{s} + \frac{1}{t}$$

is maximal subject to the existence of elements x, y and z generating G, with

$$x^r = y^s = z^t = xyz = 1.$$

The symmetric genus of each of the sporadic simple groups is discussed in [2], where a table of results shows that the symmetric genus is now known in 23 of the 26 cases. The remaining three are Fi_{23}, B and M. The case of the Baby Monster was dealt with in [5]. In this paper we deal with the case Fi_{23}.

The proof is entirely computational, but we aim to provide enough information here to enable anyone with generators for the group to be able to reproduce our result. We use the notation of the ATLAS [1] for conjugacy classes.

First note that Kleidman, Parker and Wilson proved that Fi_{23} is not a Hurwitz group [3], and we recall from [2] that generators for Fi_{23} of the above form with $(r, s, t) = (2, 3, 9)$ are already known. Now it is easy to see that the latter gives rise to a smaller genus than any generating set with $n = 4$, and so, as already noted in [2], the only cases left to consider are $(r, s, t) = (2, 3, 8)$ and $(r, s, t) = (2, 4, 5)$. The first of these gives a larger value of $\frac{1}{r} + \frac{1}{s} + \frac{1}{t}$, so we look at this case first.

We start with the 'standard generators' for Fi_{23} defined in [6]. These are elements $a \in 2B$ and $b \in 3D$ with ab of order 28. Such generators can easily be found, as a random pair of a $2B$-element and a $3D$-element has a probability greater than 1 in 60 of being conjugate to (a, b). We work with the permutations on 31671 points, as this is the fastest representation to compute in.

Next we find an element in class $2C$, such as $c = (ab(abab^2)^2)^6$. Then we make 'random' elements in $2C$ by conjugating c by 'random' group elements, and similarly we make 'random' elements in $3D$ by conjugating b. Very soon we find a pair $(d = (ab)^{-8}c(ab)^8, e = (ab^2)^{-12}b(ab^2)^{12})$ with the property that de has order 8, and $\langle d, e\rangle$ is transitive on the 31671 points. Now it is easy to see, for example from [3], that there is no transitive proper subgroup of Fi_{23}, since it would have to have order divisible by both 17 and 23. Thus $\langle d, e\rangle = Fi_{23}$, and we have found generators as above with $(r, s, t) = (2, 3, 8)$. In fact we may also note that de has just 3 fixed points in the permutation representation, and therefore is in class $8C$.

These generators give the maximal value of $\frac{1}{r} + \frac{1}{s} + \frac{1}{t}$, namely $\frac{23}{24}$, and we can calculate the symmetric genus of Fi_{23} as

$$g = 1 + \frac{1}{2}|Fi_{23}|(1 - \frac{1}{2} - \frac{1}{3} - \frac{1}{8}) = 85197301526937601.$$

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References

[1] J. H. Conway, R. T. Curtis, S. P. Norton, R. A. Parker and R. A. Wilson, *An ATLAS of Finite Groups*, Oxford University Press, 1985.

[2] M. D. E. Conder, R. A. Wilson and A. J. Woldar, The symmetric genus of sporadic groups, *Proc. Amer. Math. Soc.*, **116** (1992), 653–663.

[3] P. B. Kleidman, R. A. Parker and R. A. Wilson, The maximal subgroups of the Fischer group Fi_{23}, *J. London Math. Soc. (2)* **39** (1989), 89–101.

[4] T. W. Tucker, Finite groups acting on surfaces and the genus of a group, *J. Combin. Theory (B)*, **34** (1983), 82–98.

[5] R. A. Wilson, The symmetric genus of the Baby Monster, *Quart. J. Math. (Oxford)* **44** (1993), 513–516.

[6] R. A. Wilson, Standard generators for sporadic groups, *J. Algebra*, to appear.