# A representation for the Lyons group in $GL_{2480}(4)$, and a new uniqueness proof

Robert A. Wilson
School of Mathematics and Statistics,
The University of Birmingham

published in Arch. Math. (Basel) 70 (1998), 11–15

**Abstract**

We show how to construct a representation of the Lyons sporadic simple group in 2480 dimensions over GF(4), which is the smallest faithful representation in characteristic 2. On the way, we construct the smallest representation of $G_2(5)$ in characteristic 2, which has degree 124. Finally, we will give a new uniqueness proof for the Lyons group.

MSC 20C34, 20D08

## 1 Introduction

This note is a companion to [6], in which a representation of the Lyons group Ly in $GL_{651}(3)$ is constructed. Much of the group-theoretical part of the present construction is the same as before, while the representation-theoretical part obviously differs. The methods are by now fairly standard, following the general procedure described in [11], and making use of the Meat-axe package [10], [12].

The smallest representation of the Lyons group in characteristic 2 is of degree 2480, and the smallest field over which this representation is realisable is GF(4). Indeed, there are two such representations, which are dual to each other, and which are simply the reductions modulo 2 of the complex irreducibles of the same degree. We construct both of these representations. It turns out also that, because our chosen representation lifts to characteristic 0, we obtain as a bonus an easy proof of the uniqueness of a group of Lyons type.

Our plan here is, as in [6], to amalgamate two subgroups $G_2(5)$ and $5^3{\cdot}L_3(5)$ over their intersection $5^{2+1+2}{:}GL_2(5)$. The representation restricts irreducibly to both $G_2(5)$ and $5^3{\cdot}L_3(5)$. The representation of the latter group can be made easily as it is contained in the same permutation representation on 7750 points which was used in [6]. The representation of $G_2(5)$ turned out not to be so easy to construct, however, so we begin by describing the construction of this subgroup.

## 2 The representation of $G_2(5)$

It turned out that the 2480-dimensional 2-modular representation could not be made in a straightforward manner from the (smallish) permutation representations which can be made by permuting sets of vectors or subspaces in the natural 7-dimensional representation over GF(5). However, it is contained in the skew-square of a 124-dimensional representation, which we decided to make from scratch. In this section we outline the strategy for the construction. Details needed in order to repeat the calculations are relegated to an Appendix.

The plan here is similar to that for constructing the Lyons group, but on a smaller scale. We amalgamate subgroups $L_3(5)$ and $5^{2+1+2}{:}GL_2(5)$ over their intersection $5^2{:}GL_2(5)$. It turns out that the representation is irreducible on restriction to $L_3(5)$, and breaks up as $4 \oplus 120$ for $5^{2+1+2}{:}GL_2(5)$. For the intersection $5^2{:}GL_2(5)$, the representation has the shape $4 \oplus 24 \oplus 96$.

The representation of $L_3(5)$ is easy to make. For example, the permutation representation on the 1-dimensional subspaces of the natural module reduces modulo 2 as $1 + 30$, and the skew-square of this 30 breaks up as $1a^3 + 30a^2 + 124ab^2$ in Atlas notation [2], [5]. In fact the representation we want has indicator $+$, so is $124a$, that is $\varphi_{13}$, in the notation of [5].

It is easy to find a subgroup $5^2{:}GL_2(5)$ in $L_3(5)$, and to verify that the representation restricts as $4 + 24 + 96$. (It does not matter which class of $5^2{:}GL_2(5)$ we take, since the outer automorphism of $L_3(5)$ interchanges them and fixes the given representation.)

To make the representation of $5^{2+1+2}{:}GL_2(5)$, we first found this group as a subgroup of $G_2(5)$ in its natural representation. The orbits of this subgroup on the 3906 isotropic 1-spaces are $1 + 30 + 750 + 3125$, and the orbit of size 750 gives rise to a 2-modular representation whose Brauer character is $1a^2 + 4a + 20a^2 + 24a + 40ab + 120a + 480a$. Thus it is easy to make a representation of shape $4 \oplus 120$. At this stage we make no attempt to prove that this is the correct representation—we simply try it and see if it works.

In fact, it does not work. We must replace $4a$ by $4b$, the unique other 4-dimensional 2-modular irreducible of $5^{2+1+2}{:}\mathrm{GL}_2(5)$, which can be chopped out of the skew-square of $4a$.

Next we find a subgroup $5^2{:}\mathrm{GL}_2(5)$ in $5^{2+1+2}{:}\mathrm{GL}_2(5)$, and adjust the generators so that they are compatible with the generators already found for $5^2{:}\mathrm{GL}_2(5)$ inside $\mathrm{L}_3(5)$. Then we use the usual standard basis method (see [6] for example) to identify these two copies of $5^2{:}\mathrm{GL}_2(5)$. Since we are working over GF(2), and all the representations considered are sums of distinct irreducibles, it is easy to see that all the groups used have trivial centralizer in $\mathrm{GL}_{124}(2)$. It follows at once that the group generated by $\mathrm{L}_3(5)$ and $5^{2+1+2}{:}\mathrm{GL}_2(5)$ intersecting in $5^2{:}\mathrm{GL}_2(5)$ is isomorphic to $G_2(5)$. (In fact, it is more useful to find two matrices $X$ and $Y$ in this group which we believe to be standard generators for $G_2(5)$, and prove that they do indeed generate a group isomorphic to $G_2(5)$. We find words in the standard generators $X, Y$ giving elements $A, B, C, C_1, \ldots, C_5, D, E$ which satisfy the relations given in [6], and then find words in these new generators which give back new standard generators, $X', Y'$. Finally we use the usual 'standard basis' algorithm to prove that the groups $\langle X, Y\rangle$ and $\langle X', Y'\rangle$ are isomorphic, and therefore are identical. It follows that all three groups are equal, and isomorphic to $G_2(5)$.)

Finally, we chop the 2480-dimensional representation of $G_2(5)$ out of the skew-square of this 124.

# 3 Constructing the Lyons group

We now have copies of $G_2(5)$ and $5^3{\cdot}\mathrm{L}_3(5)$, each acting irreducibly on a 2480-space over GF(2). To construct the Lyons group, we first extend the field to GF(4). Next, to avoid duplication of work, we find 'standard' generators for $G_2(5)$, that is, generators conjugate to those used in the earlier construction [6] (see the Appendix for a definition of these standard generators). Then all the words found earlier for subgroup generators can be re-used.

We find that the 2480 restricts to $5^{2+1+2}{:}\mathrm{GL}_2(5)$ as $80a+240ab+480abcd$. A straightforward counting argument shows that there are now $3^6$ cases to consider. All but two of these are easily eliminated. To prove that the remaining two are isomorphic to the Lyons group, it suffices to follow the recipe given in [6] for verifying that the Sims presentation is satisfied. (Note that each of these representations is the dual of the other.)

# 4 Uniqueness of the Lyons group

No straightforward proof of the uniqueness of a 'group of Lyons type' appears to exist in the literature. The recent hand proof by Aschbacher and Segev [1] is very long and complicated, and all the published computer proofs, including ours, ultimately rely on Sims' construction [14], which proves both existence and uniqueness. On the other hand, the recent paper by Cooperman *et al.* [3] describes the calculation of permutations on 9606125 points, which has now been completed to a computational existence proof of the Lyons group by Gollan [4], using some of Sims' techniques, and enormous computer power. A new, more-or-less computer-free, proof of existence and uniqueness is given by Meierfrankenfeld and Parker [8], but is again rather long and technical.

Now Lyons [7] already calculated the character table, and showed that $G_2(5)$ is a subgroup of any group of Lyons type. The 5-local subgroups were completely classified in [15], from which it is easy to see that there is a subgroup $5^3{\cdot}\mathrm{L}_3(5)$ which intersects $G_2(5)$ in $5^{2+1+2}{:}\mathrm{GL}_2(5)$. The two ordinary characters of degree 2480 can be restricted to these two maximal subgroups, and all the restrictions are easily seen to be irreducible.

The 2-modular characters of the subgroup $G_2(5)$ are known (they are included for example in GAP 3.4 [13]), and in particular it is known that the representation of degree 2480 remains irreducible on reduction modulo 2. Thus there are two irreducible 2-modular characters of the Lyons group of degree 2480, whose values are given by restricting the ordinary characters to the 2-regular classes.

Our construction shows that there are at most two ways of constructing these two 2-modular representations of Ly (that is, one for each), and so it follows that the Lyons group is unique. Notice that this does not rely on the existence of the Lyons group, and does not require the checking of the Sims presentation carried out at the end of the previous section. It does however rely on our proof that the group constructed in section 2 really is $G_2(5)$, and also on the fact that the group we claim is $5^{3\cdot}L_3(5)$ really has that shape. The latter is however a relatively straightforward calculation: for example, GAP [13] tells us that the permutations on 7750 points generate a group of order 46500000, which is the order of $5^{3\cdot}L_3(5)$. Then we can use condensation methods to help chop up the permutation module modulo 5, and find a 10-dimensional constituent, whose symmetric square contains a 3-dimensional constituent, on which the whole of $L_3(5)$ is represented. Finally, the existence of (non-central) elements of order 25 in our group shows that it is a non-split extension of the natural module by $L_3(5)$.

Thus we have a new proof of uniqueness of Ly, assuming only the facts proved by Lyons in his original paper [7] and a little more 5-local information proved in [15].

ACKNOWLEDGEMENTS

I would like to thank the Science and Engineering Research Council for a grant under their Computational Science Initiative to purchase a computer on which most of these calculations were performed. I am also grateful to Chris Parker for helpful comments, in particular for asking whether one could obtain existence or uniqueness from the above construction.

# References

[1] M. Aschbacher and Segev, The uniqueness of groups of Lyons type, *J. Amer. Math. Soc.* **5** (1992), 75–98.

[2] J. H. Conway, R. T. Curtis, S. P. Norton, R. A. Parker and R. A. Wilson, *An ATLAS of Finite Groups*, Clarendon Press, Oxford, 1985.

[3] G. Cooperman, L. Finkelstein, M. Tselman and B. York, Constructing permutation representations for large matrix groups, in *Proc. ISSAC-94*. ACM Press, 1994.

[4] H. Gollan, A new existence proof for the Lyons group, Preprint, 1997.

[5] C. Jansen, K. Lux, R. A. Parker and R. A. Wilson, *An Atlas of Brauer Characters*, Clarendon Press, Oxford, 1995.

[6] C. Jansen and R. A. Wilson, The minimal faithful 3-modular representation for the Lyons group, *Comm. Algebra* **24** (1996), 873–879.

[7] R. Lyons, Evidence for a new finite simple group, *J. Algebra* **20** (1972), 540–569. Erratum, *ibid.* **34** (1975), 188–189.

[8] U. Meierfrankenfeld and C. Parker, A computer-free construction of the Lyons group, *in preparation.*

[9] W. Meyer, W. Neutsch and R. A. Parker, The minimal 5-representation of Lyons' sporadic group, *Math. Ann.* **272** (1985), 29–39.

[10] R. A. Parker, The computer calculation of modular characters (The 'Meat-axe'), in *Computational Group Theory (ed. M. D. Atkinson)*, Academic Press, 1984, pp. 267–274.

[11] R. A. Parker and R. A. Wilson, Computer construction of matrix representations of finite groups over finite fields, *J. Symbolic Comput.* **9** (1990), 583–590.

[12] M. Ringe, *The C Meat-axe 2.3, documentation*, RWTH Aachen, 1995.

[13] M. Schönert et al., *GAP 3.4 Manual (Groups, Algorithms, and Programming)*, RWTH Aachen, 1994.

[14] C. C. Sims, The existence and uniqueness of Lyons' group, in *Finite Groups '72 (eds. T. Gagen, M. P. Hale and E. E. Shult)*, North-Holland, 1973, pp. 138–141.

[15] R. A. Wilson, The subgroup structure of the Lyons group, *Math. Proc. Cambridge Philos. Soc.* **95** (1984), 403–409.

[16] R. A. Wilson, Standard generators for sporadic simple groups, *J. Algebra* **184** (1996), 505–515.

# Appendix

Here we give the details of the calculations, which should be sufficient to enable the construction to be repeated without too much difficulty.

First we take the group $L_3(5)$ to be generated by the two matrices

$$a = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ and } b = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

Then the subgroup $5^2{:}\mathrm{GL}_2(5)$ may be generated by $c = ((ab)^3b)^6$ and $d = (ab^2)^{-1}(ab)^2(abab^2)^2ab^2$.

Next we define standard generators $e$ and $f$ for $G_2(5)$ as in [6], that is $e \in 2A$, $f \in 3B$, with $ef$ of order 7 and $[e, f]$ of order 15. We find a subgroup $5^{2+1+2}{:}\mathrm{GL}_2(5)$ using the same words as in [6]. That is, put

$$\begin{aligned} g &= efef^2(ef)^5fef(ef^2)^3ef(ef^2)^2efef^2ef \\ h &= (ef)^{-5}g^6(ef)^5 \\ i &= (gefef^2)^2 \end{aligned}$$

so that $h$ and $i$ generate $5^{2+1+2}{:}\mathrm{GL}_2(5)$.

Inside this group we find $5^2{:}\mathrm{GL}_2(5)$ to be generated by $j = (hihi^2)^4$ and $l = k^{-2}hik^2[[k, j], j]$, where $k = (hihi^2)^{10}(hi)^{12}$. We next need to change generators for this group, by putting $m = ((lj)^3(ljlj^2)^2(lj^2)^2)^2$ and $n = (lj)^2(ljlj^2)^2lj^2$, and then replacing $m$ by $o = ((mn)^{12}m)^5$.

We then find that the generating pair $(o, n)$ for the second group $5^2{:}\mathrm{GL}_2(5)$ is equivalent to $(c, d)$ for the first. If $x$ is a matrix which conjugates $(o, n)$ to $(c, d)$, then we may take $G_2(5)$ to be generated by $p = ab$ and $q = x^{-1}hix$.

Next we have to find standard generators for this new copy of $G_2(5)$. To do this, set

$$\begin{aligned} r &= ((pq)^2(pqpq^2)^2)^{-4}p^{12}((pq)^2(pqpq^2)^2)^4 \\ s &= (pq^2)^{-6}(pq)^{10}(pq^2)^6. \end{aligned}$$

It turns out that $(r, s)$ is then a pair of standard generators for $G_2(5)$ in the above sense.

All other words required are given in [6]. (Incidentally, Gollan tells me that he has checked all these words, and that they all do what we claim, except that $h_{10}$ does not centralize $db^2$. The reason is that the centralizer of $db^2$ is actually $8 \circ 2A_5$ rather than $8 \circ 2S_5$ as stated in [6], and $h_{10}$ in fact conjugates $db^2$ to its fifth power. This does not affect any of the calculations.)