The McKay conjecture is true for the sporadic simple groups

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Abstract

The McKay conjecture states that the number of irreducible complex characters of a group G which have degree prime to p, is equal to the same number for the Sylow p-normalizer in G. We verify this conjecture for the 26 sporadic simple groups.

1 Introduction

McKay's conjecture in its simplest form states that if G is a finite group, p a prime, and P a Sylow p-subgroup of G, then the number of irreducible complex characters of G whose degree is not divisible by p is equal to the number of irreducible complex characters of $N_G(P)$ with the same property. This conjecture is related to the Alperin and Dade conjectures (see for example [5]), and all three seem to be true for very deep reasons, though a proof may still be a long way off. It is known to be true in the case when P is cyclic—this follows from the extensive theory of Brauer trees—see [7].

The work described in this paper was mostly done well over ten years ago, but not written up at the time, partly due to difficulties in completing one or two tricky cases. Some of the results were obtained earlier by Ostermann [10], as corollaries of his calculations of the character tables of many of the Sylow

p-normalizers of sporadic groups. Our proofs here, however, use far less information about the Sylow p-normalizers than this. Again, some of the results have since been obtained independently by Jianbei An and others, as part of the much more general verification of the Dade conjectures for the smaller sporadic simple groups (see for example [2], [1],[8], [9], etc.). Nevertheless, we considered it worth while to present all these results again, both for completeness and because our proofs are often simpler.

In what follows, the word 'character' will always mean an irreducible complex character.

In general, we note that the degrees of the characters of P are powers of p, so are either divisible by p, or equal to 1. In the latter case, they are lifts of characters of P/P'. It follows immediately, by Clifford's theorem, that all characters of $N_G(P)$ which have degree coprime to p arise in the same way as lifts of characters of $N_G(P)/P'$. On the other hand, a closer inspection of the character table of $N_G(P)/P'$, using Clifford theory again, shows that every character has degree dividing $|N_G(P)/P|$, which is prime to p. Thus our task is reduced to counting the number of characters (or equivalently, conjugacy classes) of $N_G(P)/P'$, for non-cyclic Sylow p-subgroups P.

Now $N_G(P)/P'$ is a split extension of an abelian p-group by a p'-group, isomorphic to $N_G(P)/P$. Clifford theory gives us a simple parametrization of the characters of a group of this shape: consider the orbits of $N_G(P)/P$ on the characters of P/P', and for each orbit pick the stabilizer I_{χ_i} of a representative point χ_i (called the *inertial group*). Then the characters of $N_G(P)/P'$ are in one-to-one correspondence with the pairs (χ_i, ξ_j) as i runs over orbits, and ξ_j runs over the characters of I_{χ_i}.

Thus the total number of characters of $N_G(P)/P'$ is equal to the sum over the orbits of $N_G(P)$ on characters χ of P/P', of the number of characters of the inertial subgroup I_χ in $N_G(P)/P$. It turns out that in all cases in the sporadic groups, P/P' is elementary abelian, though this is not always immediately obvious. This slightly simplifies some of the calculations.

We first present a table of results (see Table 1), showing the structure of $N_G(P)/P'$, and the number of its conjugacy classes, in each case. In later sections we present some examples of the proofs.

Table 1: Results

G	p	$N_G(P)/P'$	No. of chars.
M_{11}	2	2^2	4
	3	$3^2{:}SD_{16}$	9
M_{12}	2	2^3	8
	3	$S_3 \times S_3$	9
J_1	2	$2^3{:}7{:}3$	8
M_{22}	2	2^3	8
	3	$3^2{:}Q_8$	6
J_2	2	$2 \times A_4$	8
	3	$3^2{:}8$	9
	5	$5^2{:}D_{12}$	14
M_{23}	2	2^3	8
	3	$3^2{:}SD_{16}$	9
HS	2	2^3	8
	3	$2 \times 3^2{:}SD_{16}$	18
	5	$5^2{:}QD_{16}$	13
J_3	2	$2 \times A_4$	8
	3	$3^2{:}8$	9
M_{24}	2	2^4	16
	3	$3^2{:}D_8$	9
McL	2	2^3	8
	3	$(3^2{:}4 \times 3){\cdot}2$	12
	5	$5^2{:}3{:}8$	13
He	2	2^4	16
	3	$3^2{:}D_8$	9
	5	$5^2{:}4A_4$	16
	7	$7^2{:}(S_3 \times 3)$	20
Ru	2	2^3	8
	3	$3^2{:}SD_{16}$	9
	5	$5^2{:}(4 \wr 2)$	20
Suz	2	$A_4 \times 2^2$	16
	3	$(3 \times 3^2{:}D_8){:}2$	18
	5	$5^2{:}(4 \times S_3)$	16
O'N	2	2^3	8
	3	$3^4{:}2^{1+4}D_{10}$	18
	7	$7^2{:}(3 \times D_8)$	20
Co_3	2	2^4	16
	3	$S_3 \times 3^2{:}SD_{16}$	27
	5	$5^2{:}24{:}2$	20

G	p	$N_G(P)/P'$	No. of chars.
Co_2	2	2^5	32
	3	$S_3 \times 3^2{:}SD_{16}$	27
	5	$5^2{:}4S_4$	20
Fi_{22}	2	2^4	16
	3	$S_3 \times 3^2{:}2$	18
	5	$5^2{:}4S_4$	20
HN	2	$2^2 \times A_4$	16
	3	$S_3 \times 3^2{:}4$	18
	5	$D_{10} \times 5{:}4$	20
Ly	2	2^3	8
	3	$S_3 \times 3^2{:}SD_{16}$	27
	5	$5{:}4 \times 5{:}4$	25
Th	2	2^4	16
	3	$\frac{1}{2}(S_3 \times S_3 \times S_3)$	15
	5	$5^2{:}4S_4$	20
	7	$7^2{:}(3 \times 2S_4)$	27
Fi_{23}	2	2^4	16
	3	$S_3 \times S_3 \times S_3$	27
	5	$5^2{:}4S_4$	20
Co_1	2	2^5	32
	3	$S_3 \times 3^2{:}SD_{16}$	27
	5	$5{:}4 \times 5{:}4$	25
	7	$7^2{:}(3 \times 2A_4)$	27
J_4	2	2^5	32
	3	$(2 \times 3^2{:}8){:}2$	18
	11	$11^2{:}(5 \times 2S_4)$	42
Fi_{24}'	2	2^5	32
	3	$S_3 \times S_3 \times 3^2{:}2$	54
	5	$(A_4 \times 5^2{:}4A_4){:}2$	56
	7	$7^2{:}(6 \times S_3)$	25
B	2	2^6	64
	3	$S_3 \times S_3 \times S_3$	27
	5	$5{:}4 \times 5{:}4$	25
	7	$(2^2 \times 7^2{:}(3 \times 2A_4))\cdot 2$	81
M	2	2^6	64
	3	$S_3 \times 3^2{:}SD_{16} \times S_3$	81
	5	$5{:}4 \times 5^2{:}(4 \times S_3)$	80
	7	$7{:}6 \times 7{:}6$	49
	11	$11^2{:}(5 \times 2A_5)$	50
	13	$13^2{:}(3 \times 4S_4)$	55

Table 2: Sylow 7-normalizers

G	P	H	$N_G(P)/P'$
He	7^{1+2}	$7^2{:}SL_2(7)$	$7^2{:}(S_3 \times 3)$
$O'N$	$7^1 + 2$	$L_3(7){:}2$	$7^2{:}(3 \times D_8)$
Th	7^2	$-$	$7^2{:}(3 \times 2S_4)$
Co_1	7^2	$-$	$7^2{:}(3 \times 2A_4)$
Fi_{24}'	7^{1+2}	$He{:}2$	$7^2{:}(S_3 \times 6)$
B	7^2	$(2^2 \times F_4(2)){:}2$	$(2^2 \times 7^2{:}(3 \times 2A_4)){\cdot}2$
M	$7^{1+4}{:}7$	$7^{1+4}{:}(3 \times 2S_7)$	$7{:}6 \times 7{:}6$

2 Proofs for large primes, $p \geq 7$

For $p = 13$, the only case which arises is the Monster, M, in which the Sylow 13-normalizer is a maximal subgroup of the shape $13^{1+2}{:}(3 \times 4S_4)$. Thus $N_G(P)/P' \cong 13^2{:}(3 \times 4S_4)$, in which the action of $3 \times 4S_4$ is given by the (unique) embedding into $GL_2(13) \cong 3 \times 2{\cdot}(2 \times L_2(13)){\cdot}2$. A straightforward calculation (using 2×2 matrices over $GF(13)$, if necessary) shows that $3\times4S_4$ has just two orbits, of lengths 72 and 96, on the 168 non-trivial characters of $P/P' \cong 13^2$, and so the inertial groups of the non-trivial characters have orders 4 and 3 respectively. Moreover, the group $4S_4$ is well understood, and has 16 characters. Thus $N_G(P)/P'$ has exactly $3\times16+4+3 = 55$ characters.

For $p = 11$, two cases arise: the Monster, in which $P' = 1$ and $N_G(P) \cong 11^2{:}(5 \times 2A_5)$, and J_4, in which $N_G(P) \cong 11^{1+2}{:}(5 \times 2S_4)$, so $N_G(P)/P' \cong 11^2{:}(5 \times 2S_4)$. In both cases, the calculations are very similar to the above.

In M, the group $5 \times 2A_5$ has just $5 \times 9 = 45$ characters, and acts transitively on the 120 non-trivial characters of P. The inertial subgroup therefore has order 5, and the total number of characters of $N_G(P)$ is $45 + 5 = 50$.

Similarly, in J_4, the group $5\times2S_4$ acts transitively on the 120 non-trivial characters of P/P', the inertial subgroup therefore has order 2, and the total number of characters of $N_G(P)/P'$ is $5 \times 8 + 2 = 42$.

In the case $p = 7$, again it turns out that in every case $P/P' \cong 7^2$, and similar calculations produce the results without too much difficulty. The cases are listed in Table 2, together with a subgroup H in which the structure of $N_G(P)$ can be easily seen.

In each case there is up to conjugacy only one subgroup of $GL_2(7)$ which acts as $N_G(P)/P$ does on P/P', and therefore it is again a simple calculation to obtain the results given in Table 1.

For $G \cong He$, there are five orbits of $N_G(P)/P$ on non-trivial characters of P/P', with lengths 6, 6, 9, 9, 18, and inertial groups of orders 3, 3, 2, 2, 1. Since $S_3 \times 3$ itself has $3 \times 3 = 9$ characters, we obtain a total of $9 + 3 + 3 + 2 + 2 = 1 = 20$ characters of $N_G(P)/P'$.

For $G \cong O'N$, $N_G(P)/P' \cong 3 \times D_8$ has $3 \times 5 = 15$ characters, and the non-trivial orbit lengths are 12, 12, 24, corresponding to inertial groups of orders 2, 2, 1. Thus $N_G(P)/P'$ has $15 + 2 + 2 + 1 = 20$ characters.

For $G \cong Th$, $N_G(P)/P' \cong 3 \times 2S_4$ has $3 \times 8 = 24$ characters, and is transitive on the non-trivial characters of P/P', with inertial group of order 3. Thus $N_G(P)/P'$ has $24 + 3 = 27$ characters.

For $G \cong Co_1$, $N_G(P)/P' \cong 3 \times 2A_4$ has $3 \times 7 = 21$ characters, and has two orbits of size 24 on the non-trivial characters of P/P', each with inertial group of order 3. Thus $N_G(P)/P'$ has $21 + 3 + 3 = 27$ characters.

For $G \cong Fi'_{24}$, we calculate similarly that there are $18 + 3 + 2 + 2 = 25$ characters of $N_G(P)/P'$, and for $G \cong M$ we have $7 \times 7 = 49$ characters.

Finally we consider the case $G \cong B$. It is easy to see that there are just two inertial groups, namely $(2^2 \times (3 \times 2A_4))^{\cdot}2$ and $2^2 \times 3$. The latter group clearly has 12 characters. To see how many characters the former group has, we need to do a little more work. First note that the normal subgroup of order 4 has 2 conjugacy classes fixed by the outer automorphism, and one pair of classes swapped by this automorphism. Similarly, the group $3 \times 2A_4$ has a total of 21 conjugacy classes, of which 9 are fixed, and six pairs swapped, by the automorphism. Thus the number of conjugacy classes in the inner half of the inertial group is $2 \times 9 + 2 \times 6 + 1 \times 9 + 2 \times 1 \times 6 = 51$. The number of classes in the outer half of $2^2{:}2 \cong D_8$ is 2, while the number in the outer half of $(3 \times 2A_4)^{\cdot}2 \cong 3 \times 2S_4$ is $3 \times 3 = 9$, giving a total count of $2 \times 9 = 18$ outer classes, and a total number of characters $51 + 18 + 12 = 81$.

3 The case $p = 5$

In the cases $G \cong J_2, He, Suz, Fi_{22}, Fi_{23}$ and Fi'_{24}, the Sylow 5-subgroup is an elementary abelian group 5^2, and the structure of its normalizer is clear. In the cases $G \cong HS, McL, Ru, Co_3, Co_2$ and Th, the Sylow group is an extraspecial group 5^{1+2}, so again $P/P' \cong 5^2$, and the calculations we are required to do are very similar.

For $G \cong J_2$, we have $D_{12} \cong 2 \times S_3$ acting, with four orbits of size 6, giving $6 + 2 + 2 + 2 + 2 = 14$ characters.

For $G \cong HS$, we have $QD_{16} = \langle a, b \mid a^8 = b^2 = 1, a^b = a^5 \rangle$ acting, with non-trivial orbit sizes 8 and 16, and with 10 conjugacy classes of elements. Thus there are $10 + 2 + 1 = 13$ characters of $5^2{:}QD_{16}$ in total.

For $G \cong McL$, we have $3{:}8 = \langle a, b \mid a^3 = b^8 = 1, a^b = a^{-1} \rangle$ acting, with a single orbit of size 24. The group 3:8 itself has 8 linear characters and 4 of degree 2, giving a total of $12 + 1 = 13$ characters of $5^2{:}3{:}8$.

For $G \cong He$, we have $4A_4$ acting, with a single orbit of size 24. The inertial groups are therefore $4A_4$ and 2, giving a total of $14 + 2 = 16$ characters. A similar argument works for the cases $G \cong Co_2, Fi_{22}, Th$ and Fi_{23}, where $N_G(P)/P' \cong 5^2{:}4S_4$, and shows that there are $16 + 4 = 20$ characters.

For $G \cong Ru$, we have $4 \wr 2$ acting, with non-trivial orbit sizes 8 and 16, giving a total of $14 + 4 + 2 = 20$ characters.

The case $G \cong Suz$ is discussed together with the Monster below.

The case $G \cong Co_3$ has a transitive group $24{:}2 \cong \langle x, y \mid x^{24} = y^2 = 1, x^y = x^5 \rangle$ acting, with an inertial group of order 2. Thus there are $18 + 2 = 20$ characters in total.

The groups Co_2, Fi_{22}, Th and Fi_{23} all have a group $5^2{:}4S_4$ for $N_G(P)/P'$. There is just one proper inertial subgroup, of order 4, and we have already seen that $4S_4$ has 16 conjugacy classes, so the total number of characters of $N_G(P)/P'$ is $4 + 16 = 20$.

Finally, for $G \cong Fi_{24}'$, the inertial groups are $(A_4 \times 4A_4){:}2$ and $S_4 \times 2$. The latter has 10 characters, while the former can be shown to have 46, giving a total of 56. To see this, note that A_4 has two classes fixed by the outer automorphism, and one pair of swapped classes. Similarly, $4{}^{\cdot}A_4$ has six fixed classes, and four pairs of swapped classes. Thus the number of inner classes is $2 \times 6 + 1 \times 6 + 2 \times 4 + 2 \times 1 \times 4 = 34$. Also, S_4 has two outer classes, while $4S_4$ has six outer classes, giving $2 \times 6 = 12$ outer classes in all.

The remaining cases (where the Sylow group has order bigger than 5^3) are listed in Table 3, along with a maximal subgroup H in which the structure of the Sylow 5-normalizer can be easily elucidated.

In the case of Co_1, the shape of $N_G(P)/P'$ is obvious. The three cases $G \cong HN, Ly$ and B all have isomorphic Sylow 5-subgroups, and we see for example from [11] that $P/P' \cong 5^2$, from which the shape of $N_G(P)/P'$ follows immediately. [Specifically, the group $4S_6$ has two orbits, of lengths 36 and 120, on the 156 one-dimensional subspaces of $5^{1+4}/5$, so a simple counting argument shows that an element of order 5 in $4S_6$ fixes just one

Table 3: The larger cases for $p = 5$

G	H	$N_G(P)$	$N_G(P)/P'$
HN	$5^{1+4}.2^{1+4}.5.4$	$5^{1+4}{:}(2 \times 5{:}4)$	$5{:}2 \times 5{:}4$
Ly	$5^{1+4}.4S_6$	$5^{1+4}{:}(4 \times 5{:}4)$	$5{:}4 \times 5{:}4$
Co_1	$5^{1+2}.GL_2(5)$	$5^{1+2}{:}(4 \times 5{:}4)$	$5{:}4 \times 5{:}4$
B	$5^{1+4}.2^{1+4}.A_5.4$	$5^{1+4}{:}(4 \times 5{:}4)$	$5{:}4 \times 5{:}4$
M	$5^{1+6}.4J_2.2$	$5^{1+6}{:}(4 \times 5^2{:}(4 \times S_3))$	$5{:}4 \times 5^2{:}(4 \times S_3)$

of these one-dimensional subspaces. Similarly, the 5-element fixes just one hyperplane, namely the orthogonal complement of the fixed 1-space.] A similar calculation in $5^{1+6}{:}4.J_2.2$ is carried out in [13], with the same result.

It is now a triviality to calculate the number of characters of $N_G(P)/P'$ in all cases except the Monster, where we see that $4 \times S_3$ has orbits $1+12+12$ on the characters of 5^2, so that $5^2{:}(4 \times S_3)$ has $4 \times 3+2+2 = 16$ characters. (This calculation also occurs for the Suzuki group.) Therefore $5{:}4 \times 5^2{:}(4 \times S_3)$ has $5 \times 16 = 80$ characters.

4 The case $p = 3$

In many ways this is the hardest case, because the structure of $N_G(P)/P'$ is sometimes quite hard to determine precisely. As in the previous cases, the key is to find a suitable subgroup H in which the structure of the Sylow p-normalizer can be seen.

The smaller cases, up to and including M_{24}, are all quite straightforward, as are the cases He and Ru. We treat the other 15 cases individually.

Taking the McLaughlin group first, we see the Sylow 3-normalizer as a subgroup of shape $3^{1+4}{:}(4{\cdot}S_3)$ inside the maximal subgroup $3^{1+4}{:}2{\cdot}S_5$, from which we deduce that $N_G(P)/P \cong Q_8$ and $N_G(P)/P' \cong (3^2{:}4 \times 3){\cdot}2$. This enables us to calculate the orbits of Q_8 on the characters of the 3^3 as $1+2+8+8+8$, with inertial groups Q_8, 4, 1, 1, 1 respectively.

Next we look at the Suzuki group, and use the two subgroups $3^5{:}M_{11}$ and $3^{2+4}{:}2{\cdot}(A_4 \times 2^2).2$ to show that $N_G(P)/P \cong SD_{16}$, and $P/P' \cong 3^3$, on which the action is given by $N_G(P)/P' \cong (3 \times 3^2{:}Q_8).2$. The inertial groups are now SD_{16}, Q_8, and three copies of the trivial group.

The case Co_3 is very similar, as the Sylow 3-normalizer is contained in a group $3^5{:}(2 \times M_{11})$, and we obtain $N_G(P)/P' \cong S_3 \times 3^2{:}SD_{16}$. Now the second factor has inertial groups SD_{16} and 2, so has $7 + 2 = 9$ characters, making $3 \times 9 = 27$ for the whole group.

In Conway's second group, the Sylow normalizer is contained in the subgroup $3^4{:}(2 \times A_6{\cdot}2^2)$ inside $U_4(3).D_8$, so again we obtain $N_G(P)/P' \cong S_3 \times 3^2{:}SD_{16}$.

The argument for the Lyons group is also very similar, since it also contains a group of the shape $3^5{:}(2 \times M_{11})$, and we obtain $N_G(P)/P' \cong S_3 \times 3^2{:}SD_{16}$ again.

In the big Conway group Co_1 we use the subgroup $3^6{:}2{\cdot}M_{12}$ to come to the same conclusion again in this group.

The Sylow 3-normalizer in the O'Nan group is the maximal subgroup of shape $3^4{:}2^{1+4}{:}D_{10}$, which is transitive on the 80 non-trivial characters of the 3^4, with inertial group therefore of order 4. Now the group $2^{1+4}D_{10}$ itself has 14 irreducible characters, made up of the four characters of D_{10}, six more characters of $2^4{:}D_{10}$, and four faithful characters. Thus the total number of characters of the Sylow 3-normalizer is $4 + 14 = 18$.

For the smallest Fischer group Fi_{22} we use the subgroup $3^{1+6}.2^{3+4}.3^2.2$, and perform explicit calculation with the generators given in [12]. This shows that the 3^2 fixes a unique hyperplane in its action on the 3^6, and therefore $N_G(P)/P' \cong S_3 \times 3^2{:}2$. The second factor has just six characters, so the whole group has 18.

For the group Fi_{23}, we can use the analogous subgroup $3^{1+8}.2^{1+6}.3^{1+2}.2S_4$, or alternatively, note that the Sylow 3-normalizer actually lies in the part of this group which is in $O_8^+(3){:}S_3$, namely $3^{1+8}{:}2{\cdot}(A_4 \times A_4 \times A_4).2.S_3$. Since the top S_3 permutes the three copies of A_4, and since the action of $2{\cdot}(A_4 \times A_4 \times A_4)$ is given by the tensor product action $SL_2(3) \otimes SL_2(3) \otimes SL_2(3)$, we pick up a factor of 3 towards P/P' from each 'layer', and therefore $N_G(P)/P' \cong S_3 \times S_3 \times S_3$.

In the Harada–Norton group we use $3^4{:}2(A_4 \times A_4).4$, in which the action is given by some version of the orthogonal group $O_4^+(3)$. In particular, the action of $2(A_4 \times A_4)$ is given by the natural action of $SL_2(3) \otimes SL_2(3)$. This means that, as usual, there is unique fixed hyperplane in the action of the Sylow subgroup on the 3^4, whence $N_G(P)/P' \cong S_3 \times 3^2{:}4$, so the total number of characters is $3 \times (4 + 2) = 18$.

The case of J_4 is actually not hard, but is worth giving in some detail as the isomorphism type of the Sylow 3-normalizer has been wrongly stated in otherwise authoritative papers. We have $N_G(P) \cong (2 \times 3^{1+2}{:}8){:}2$, and

$$N_G(P)/P \cong (2{\times}8){:}2 \cong \langle x, y, z \mid x^2 = y^8 = z^2 = [x,y] = [x,z] = 1, y^z = xy^5 \rangle.$$

The latter group has just two orbits, of lengths 1 and 8, on the 9 characters of $P/P' \cong 3^2$, and moreover has 14 conjugacy classes, so the total number of characters of $N_G(P)/P'$ is $14 + 4 = 18$.

In the largest Fischer group $Fi_{24}{}'$ we use the subgroup $3^{2+4+8}(A_5{\times}2{\cdot}A_4).2$. Now it is clear that we can factor out the normal 3^6, and we are left with the group $A_5{\times}2{\cdot}A_4$ acting on 3^8 as the tensor product of the deleted permutation representation of A_5 with the natural representation of $SL_2(3)$. Thus we pick up a contribution of 3^2 to P/P' from the 3^8, and calculate $N_G(P)/P' \cong S_3 \times S_3 \times 3^2{:}2$.

In the Baby Monster, a similar calculation in the subgroup $3^{2+3+6}(S_4 \times 2{\cdot}S_4)$, in which the action on the 3^6 is given by $O_3(3) \otimes GL_2(3)$, shows that $N_G(P)/P' \cong S_3 \times S_3 \times S_3$.

Similarly in the Monster we use $3^{2+5+10}(M_{11} \times 2S_4)$, in which the action of $M_{11} \times 2S_4$ on 3^{10} is the tensor product of a 5-dimensional representation of M_{11} and the natural representation of $GL_2(3)$. Thus there is a unique fixed hyperplane, and the usual argument shows that $N_G(P)/P' \cong S_3 \times S_3 \times 3^2{:}SD_{16}$, and there are $3 \times 3 \times 9 = 81$ characters in total.

Finally we are left with the Thompson group. Here we use Aschbacher's description of the structure of the 3-local subgroups, in Chapter 14 of [3]. He states explicitly that $P/P' \cong 3^3$, and we deduce from his other results that the action of $N_G(P)/P \cong 2^2$ on it is the sum of the three non-trivial irreducibles. This implies that the orbits of 2^2 on the 27 characters of 3^3 are one of size 1 (the trivial character), three of size 2 (the fixed points of the three involutions), and 5 of size 4 (the rest). Thus we obtain a total of $4 + 3 \times 2 + 5 \times 1 = 15$ characters.

5 The case $p = 2$

In this case, the groups $N_G(P)/P'$ all have very straightforward structure, so the number of characters can be very easily determined. Moreover, it is easy to see that in all but five cases, P is self-normalizing. The exceptions are J_1, where $N_G(P)/P \cong 7{:}3$, and J_2, J_3, Suz and HN, where $N_G(P)/P \cong 3$. The only real problem, therefore, is in certain cases to work out the order of P/P'.

In M_{11}, the Sylow 2-subgroup is $SD_{16} = \langle a, b \mid a^8 = b^2 = 1, a^b = a^3\rangle$ and $P/P' \cong 2^2$. In J_1, the Sylow 2-subgroup is elementary abelian, and $N_G(P)/P' \cong 2^7{:}7{:}3$. In M_{22}, M_{23} and McL, the Sylow 2-normalizer is a subgroup $2^4{:}D_8$ of $2^4{:}A_6$, so we have $N_G(P)/P' \cong 2^3$. Now using $2^{10}{:}M_{22}$ in Fi_{22}, or its double cover in Fi_{23}, we see that the Sylow 2-subgroup fixes a unique hyperplane in the 2^{10}, and therefore P/P' has order 2^4.

In M_{24} and the Held group, $N_G(P) \cong 2^6{:}(2 \times D_8) < 2^6{:}3S_6$, in which it is easily seen using the hexacode that there is a unique hyperplane in the 2^6 fixed by the Sylow 2-subgroup, so that $N_G(P)/P' \cong 2^4$. Now using the subgroups of shape $2^{11}M_{24}$ in Co_1, J_4 and Fi_{24}' we see that the Sylow 2-subgroup of M_{24} fixes a unique hyperplane in the 2^{11}, so that $N_G(P)/P'$ has order 2^5. Similarly, using $2^{1+24}Co_1$ in the Monster we get $N_G(P)/P'$ of order 2^6.

In J_2 and J_3, the subgroup $2^{1+4}{:}A_5$ shows us that $N_G(P)/P' \cong 2 \times A_4$. In M_{12}, the Sylow normalizer $4^2{:}2^2$ is seen inside $4^2{:}D_{12}$, from which it is clear that $N_G(P)/P' \cong 2^3$.

In the Higman–Sims group HS, the Sylow normalizer $4^3{:}D_8$ is seen inside $4^3{:}L_3(2)$, which shows that $N_G(P)/P'$ is isomorphic to that in the affine group $AGL_3(2) \cong 2^3{:}L_3(2)$. This is well known to be 2^3. A similar argument produces the same answer in the O'Nan group, using the subgroup $4^3{\cdot}L_3(2)$.

In the Rudvalis group Ru we use the subgroup $2^{3+8}L_3(2)$, and the normal 2^3 is obviously contained in P', so we can quotient it out. Thus we look at $2^8{:}D_8$ inside $2^8{:}L_3(2)$, where the action of $L_3(2)$ on 2^8 is the adjoint representation (i.e. the Steinberg module). This is well known to restrict to the regular representation of D_8, and so $N_G(P)/P' \cong 2^3$.

In the Suzuki group, we use $2^{4+6}{:}3{\cdot}A_6$, or rather its quotient group $2^6{:}3{\cdot}A_6$. The 2^6 is really a 3-dimensional $GF(4)$-module for $3{\cdot}A_6$, in which the Sylow 2-normalizer $3 \times D_8$ fixes a unique hyperplane, so $N_G(P)/P' \cong A_4 \times 2^2$.

In Co_3 we can use the involution centralizer $2{\cdot}S_6(2)$. In the quotient $S_6(2)$, the Sylow 2-normalizer is contained in $2^5{:}S_6$, the action of S_6 on 2^5 being that on the space of even subsets of 6 letters. Direct calculation shows that $N_G(P)/P' \cong 2^4$.

In the Lyons group, the Sylow 2-normalizer is in $2{\cdot}A_{11}$, so in $2{\cdot}S_8$. Now in S_8, we have Sylow 2-normalizer $2^4{:}D_8 < S_4 \wr 2$, and by calculation $N_G(P)/P' \cong 2^3$.

In Co_2 we use the subgroup $2^{10}{:}M_{22}{:}2$, and first determine the image of P/P' in $M_{22}{:}2$, using the subgroup $2^4{:}S_6 \cong 2^4{:}S_4(2)$. Here we have the Sylow 2-subgroup $2 \times D_8$ in $S_4(2)$, giving a contribution of 2^3 to P/P', and we pick up another factor of 2 from the normal subgroup 2^4. Similarly, in $2^{10}{:}M_{22}{:}2$ the Sylow 2-subgroup fixes a unique hyperplane in the 2^{10}, so we pick up another factor of 2, giving $P/P' \cong 2^5$ in Co_2.

Similarly for the Baby Monster we can use the subgroup $2^{2+10+20}{:}(M_{22}{:}2\times S_3)$, and as usual work just in the quotient group $2^{20}{:}(M_{22}{:}2 \times S_3)$. Again we have a contribution of 2^5 to P/P' from the quotient $M_{22}{:}2 \times S_3$, and the representation on 2^{20} is the tensor product of a 10-dimensional representation of $M_{22}{:}2$ and the 2-dimensional representation of S_3. Thus there is again a unique fixed hyperplane for the Sylow 2-subgroup, and so in B we have P/P' of order 2^6.

Next we consider the Thompson group. Here the argument is a little trickier, as both the useful 2-local subgroups $2^5{\cdot}L_5(2)$ and $2^{1+8}{\cdot}A_9$ are non-split extensions. First we use the involution centralizer, and work first in the quotient A_9. Here the Sylow 2-normalizer is isomorphic to that in $A_8 \cong L_4(2)$, so $P/P' \cong 2^3$. Now in the action on 2^8 there is a unique fixed hyperplane, so in Th we have P/P' of order at most 2^4. On the other hand, in $L_5(2)$ we see P/P' of order at least 2^4.

Finally we consider the Harada–Norton group HN. Again we need to use two different 2-local subgroups in order to obtain the required information without going into too much detail in one particular group. We use the involution centralizer $2^{1+8}{\cdot}(A_5 \times A_5){:}2$, in which the $A_5 \times A_5$ acts on the 2^8 as $O_4^+(4)$, i.e. as $L_2(4) \otimes L_2(4)$. Hence P' contains at least a 2^5 out of the 2^{1+8}, and moreover we know that the normalizer of this 2^5 in HN has the shape $2^5(2^6{:}(L_3(2) \times 3))$. In the latter group, the $L_3(2)$ acts on 2^6 as two copies of the natural module, so we have $N_G(P)/P' \cong 2^2 \times A_4$.

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