A new computer construction of the Monster using 2-local subgroups

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Abstract

We describe a construction of the Monster simple group implicitly as 196882 × 196882 matrices over the field of 3 elements.

1 Introduction

The Monster is the largest of the 26 sporadic simple groups. Evidence for its existence emerged in the early 1970s, but due to its enormous size it could not be constructed by computer. The first construction, therefore, by Griess in 1980 [3], was entirely by hand. Implicitly, Griess's construction describes the group by generators given as 196884 × 196884 matrices over $\mathbb{Z}[1/2]$. It is therefore possible, in principle at least, to reduce the matrix entries modulo 3 to obtain a representation of the group over $GF(3)$. As there is a fixed vector and a fixed hyperplane, this gives rise to an irreducible representation of degreee 196882. In practice, however, such matrices would be useless for computation, as they occupy nearly 8GB each and it would take months or years to multiply two matrices.

We will describe a computational construction of the Monster sporadic simple group as a group generated by three 196882-dimensional matrices over $GF(3)$ and stored in a compact format which allows effective computation. The construction is based on Griess' strategy [3] and Conway's simplification of it [2] by using the 2-local structure of the Monster. It also uses many ideas and techniques from the 3-local construction over $GF(2)$ given in [5] and the construction of the Baby Monster over $GF(3)$ in [7].

The basic strategy (following Griess) is to construct the restriction of the representation to an involution centraliser $2_+^{1+24}\mathrm{Co}_1$, then restrict further to the centraliser $2^{2+11+22}{}^{\cdot}\mathrm{M}_{24}$ of two commuting involutions, and finally to adjoin an element of order 3 normalising the central 2^2 of the latter group.

We denote modules by their degrees, followed where necessary by a distinguishing letter, in boldface type. The only exceptions to this will be the Leech Lattice representation reduced modulo 2 and the permutation representation of Co_1 on 98280 points, which will be denoted by **24f2** and **P** respectively. The group for which this is a module will always be clear from the context. An index of notation is provided at the end of the paper.

Section 2 describes the construction of the group $G \cong 2_+^{1+24}{}^{\cdot}\mathrm{Co}_1$, the centraliser of an element x in class $2B$ of the Monster. We will construct generators c and d of G. They will have orders 4 and 6 respectively and will be preimages of standard generators of Co_1 as defined in [10]. The restriction of **196882**, the module for the Monster, to G is a direct sum of three modules

$$\mathbf{298} \oplus \mathbf{98280} \oplus \mathbf{98304} \cong \mathbf{298} \oplus \mathbf{98280} \oplus (\mathbf{24} \otimes \mathbf{4096}),$$

where the modules **298**, **98280** and **98304** are modules for G and the modules **24** and **4096** are modules for the double cover, $2G$.

Let y be a non-central element of $O_2(G)$ such that $\langle x, y\rangle$ is a four-group of type BBB (i.e. all its involutions are in class $2B$). We will find generators for $K = C_G(y) \cong 2^{2+11+11+11}{}^{\cdot}\mathrm{M}_{24}$ as words in c and d. Restricting **196882** to K gives a module of structure

$$\mathbf{276a} \oplus \mathbf{22} \oplus \mathbf{276b} \oplus \mathbf{276c} \oplus \mathbf{48576} \oplus \mathbf{49152a} \oplus \mathbf{49152b} \oplus \mathbf{49152c}.$$

We will define a standard basis for this module relative to a list of standard generators (a, b, u, v, w) for K and construct a basis change matrix B which will map our original basis to the standard one.

Next we construct our extra generator as a matrix T realising an automorphism σ of K of order 3. The group $\langle K, T\rangle$ will be called H. The automorphism σ may be defined by $\sigma : (a, b, u, v, w) \mapsto (a, b, w, u, v)$. In particular, T permutes the three 276-dimensional constituents of the module, the three 49152-dimensional ones, and three subgroups of K of shape 2^{2+11}. There are several matrices which realise the automorphism σ and we will need to test all the cases to select the correct one to generate the Monster. This will require a method of calculating in these groups, which will be described in Section 7.

Most of the programs used were written by the first author, based on programs described in [5]. In addition, the Meataxe [6], [8] and Magma [1] were used. This paper forms part of the first author's PhD thesis, written under the supervision of the second author.

1.1 Standard basis algorithms

In [5], the standard basis algorithm together with its variants and uses was given in great detail. Here we summarise the algorithm and state how it is used in our construction.

The matrix group version of the algorithm works as follows. The input is a set of generators of the group and a "seed" vector. The seed vector is one that can be specified precisely, *e.g.* the fixed vector of a subgroup. This seed is taken as the first vector in the basis and the other basis vectors are defined to be its images under some list of words in the group generators.

We use a version of the standard basis algorithm in Section 2.1 to construct the 4096-dimensional representation of a non-split extension $2^{1+24 \cdot}\mathrm{Co}_1$, in Section 4 to find standard generators for $O_2(K)$ and in Sections 3.2 and 5 to rewrite our generators for G with respect to a nicer basis. Note that, modulo permutations and sign changes on the vectors, the basis that we use is essentially the same as in [3] and [2]. Variants of the algorithm are also used to construct an outer element of the split extension $H \cong K{:}3$.

2 A 196882 dimensional representation of $2_+^{1+24 \cdot}\mathrm{Co}_1$

2.1 The non-monomial summands of the representation

We will construct each of the modules **298**, **98280**, **24** and **4096** for the group $2G \cong 2^{\cdot}2_+^{1+24 \cdot}\mathrm{Co}_1$ separately.

If x is an element of G we write $x_{\mathbf{m}}$ for the action of x on the module $\mathbf{m}$. We also abuse notation a little by defining $x_{\mathbf{4096}}$ as "the image of some element x' in **4096**, where x' is a preimage of x in $2G$" and similarly for $x_{\mathbf{24}}$. In this situation, we can change the sign of both $x_{\mathbf{24}}$ and $x_{\mathbf{4096}}$ simultaneously, but not separately.

The module **24** is the Leech Lattice representation of $2\mathrm{Co}_1$ reduced modulo 3, and $c_{\mathbf{24}}$ and $d_{\mathbf{24}}$ are standard generators [9]. We obtain **298** by removing one trivial submodule and one trivial quotient module from the symmetric square of **24** using the Meataxe "chop" program [6].

Let $E = 2^{1+24}$, $\bar{E} = E/Z(E)$ and **24f2** be the 24-dimensional Leech Lattice representation of $2\mathrm{Co}_1$ reduced modulo 2. The vector space $V = GF(2)^{24}$ is isomorphic to $\bar{E}$ so they can be identified with each other. The representation **4096** can be constructed by finding two matrices $c_{\mathbf{4096}}$ and $d_{\mathbf{4096}}$ which act by conjugation on generators for the natural representation of E in the same way (modulo -1) as $c_{\mathbf{24f2}}$ and $d_{\mathbf{24f2}}$ act on a basis of V.

To construct this representation we follow the method given in [5] and [7], changing the numbers where necessary. As in [7], we choose our generators for E to be a set of 24 involutions $\{g_i\}$ such that $g_i g_j = -g_j g_i$ if $i \equiv j \pmod{12}$, and $g_i g_j = g_j g_i$ otherwise. We define the basis vectors $\{v_i\}$ of V to be their images in $\bar{E}$.

2.2 The degree 98280 monomial representation of $2^{24\cdot}\mathrm{Co}_1$

This construction also follows the method of [7], but we include the details here as they were not given in this reference.

Abstractly, the monomial representation of degree 98280 is the representation induced from the non-trivial linear representation of $2^{24\cdot}\mathrm{Co}_2 = 2^{23\cdot}(2 \times \mathrm{Co}_2)$. We can obtain it via the action by conjugation of G on a certain conjugacy class of elements of E, which fall into 98280 pairs under multiplication by the central involution. We label the elements of each pair "+" and "−". So **98280** can be constructed from **4096**. Each pair corresponds to a vector in a certain orbit of $\bar{E} \cong V$, which is isomorphic to the reduction modulo 2 of the Leech Lattice under Co_1. We first construct the permutation representation of Co_1 on the 98280 points which will tell us where the nonzero entries lie in the monomial matrices generating $2^{24\cdot}\mathrm{Co}_1$, and then determine the sign of each entry.

As in [7], we use a spanning tree algorithm to provide an efficient ordering in which to calculate the entries.

Let $\Lambda = \{\lambda_1, \ldots, \lambda_{98280}\}$ be the orbit of 98280 vectors in V under the action of G. The action of $c_{\mathbf{24f2}}$ and $d_{\mathbf{24f2}}$ on Λ gives a permutation representation of c and d on the 98280 pairs of elements. We must now determine their action on the elements of each pair.

Define the sign of an element of E to be the sign of the nonzero entry ± 1 in the first row. For any given $x \in G$, suppose $\lambda_i x_{\mathbf{4096}} = \lambda_j$. Then $e_i^{x_{\mathbf{4096}}} = e_j$, where e_i and e_j are lifts of λ_i and λ_j into E. The i,j-th entry of $x_{\mathbf{98280}}$ will be $+1$ if e_i and e_j have the same sign, and -1 otherwise.

So in theory the monomial matrices could be written down simply by conjugating each element of E by x and noting whether the sign is preserved. The problem with this approach is in the implementation, as conjugating 4096-dimensional matrices 98280 times for each generator would require too much time.

Fortunately all the necessary information is available in **24f2**, and in the next section we show how to extract the information.

2.3 Implementing the construction of 98280

Fix some $\lambda \in \Lambda$ and $x \in G$. Suppose

$$\lambda = \sum_{i=1}^{24} \alpha_i v_i \qquad \text{and} \qquad \lambda x_{\mathbf{24f2}} = \sum_{i=1}^{24} \beta_i v_i,$$

where $\alpha_i, \beta_i \in \{0,1\}$.

One lift of λ into E is

$$e = \prod_{i=1}^{24} g_i^{\alpha_i} \qquad \text{so} \qquad e x_{\mathbf{4096}} = (-1)^{\varepsilon} \prod_{i=1}^{24} g_i^{\beta_i},$$

where $(-1)^{\varepsilon}$ is the sign we need to calculate.

By collecting together all the terms in $\{g_i, g_{i+12}\}$, for each i, and using the fact that $g_i^2 = 1$, we can find

$$e x_{\mathbf{4096}} \quad = \quad \prod_{i=1}^{24} (g_i x_{\mathbf{4096}})^{\alpha_i} \quad = \quad \prod_{i=1}^{12} (\ldots g_i g_{i+12} g_i g_{i+12} g_i \ldots)$$

where each of the terms $(\ldots g_i g_{i+12} g_i g_{i+12} g_i \ldots)$ may start and finish either with g_i or g_{i+12}. We can convert this to

$$e x_{\mathbf{4096}} \quad = \quad (-1)^{\varepsilon} \prod_{i=1}^{12} g_i^{\beta_i} g_{i+12}^{\beta_{i+12}}$$

by replacing each occurrence of $g_{i+12} g_i$ by $g_i g_{i+12}$ and changing sign each time we do so. So ε is the number of these operations required modulo 2. We can count these operations as follows.

Let $R{=}(r_{i,j})$ and $S{=}(s_{i,j})$ $(1 \leq i \leq 24,\ 1 \leq j \leq 12)$ be the matrices with entries $r_{i,j}{=}x_{\mathbf{24f2}\,i,j}$ and $s_{i,j}{=}x_{\mathbf{24f2}\,i,j+12}$. In other words, R consists of the first 12 columns of $x_{\mathbf{24f2}}$ and S the remaining 12. Now let $M' = (m'_{i,j}) = RS^t$, and $M = (m_{i,j})$ be the matrix with (i,j)-th entry $m'_{i,j}$ for $i < j$, 0 otherwise.

Then $m_{i,j}$ is equal to the number of "bad" pairs (products of two generators which must be interchanged) contributed to the above product by the summands v_i, v_j of λ when α_i and α_j are both non-zero. So we can count the number of such bad pairs which appear in λ^x by calculating $\lambda M \lambda^t$, as the premultiplication gives the sum of the rows i for $\alpha_i = 1$, and the postmultiplication deletes the unnecessary columns and sums all other entries.

We have now determined the signs to attatch to the permutations generating Co_1, in order to convert them to our desired representation of $2^{24\cdot}\mathrm{Co}_1$. This completes our construction of the module **196882** restricted to $G \cong 2_+^{1+24\cdot}\mathrm{Co}_1$.

3 The subgroup $K \cong 2^{2+11+22\cdot}\mathrm{M}_{24}$

3.1 Restriction of 196882 to K

Here we will find K as a subgroup of $G \cong 2^{1+24\cdot}\mathrm{Co}_1$. The crucial point is to find generators for K with the property that we can easily write down the action of the required automorphism σ on the given generators. In this section the generators are only defined up to certain ambiguities, which are resolved in Section 4. While finding the generators that we need, we will construct the monomial part of the basis change matrix $B = B_{\mathbf{196882}}$.

We shall define a standard generating set of five generators $\{a, b, u, v, w\}$ for $K = C_M(2^2) = 2^{2+11+11+11\cdot}\mathrm{M}_{24}$. At this stage u, v, and w are only defined up to certain ambiguities which will be resolved later on.

First, we will find a and b generating a subgroup $2^{11\cdot}\mathrm{M}_{24}$ (this subgroup is unique up to conjugacy). They will be preimages for standard generators of M_{24} and have orders 4 and 3 respectively where ab has order 23 and $ab(abab^2)^2ab^2$ has order 4. It is not necessary to specify these generators more precisely.

Let $\bar{K} = K/2^{2+11} \cong (2^{11} \times 2^{11}){:}\mathrm{M}_{24}$. Then $\bar{K}$ has three normal subgroups of order 2^{11}, which we call $\bar{U}$, $\bar{V}$ and $\bar{W}$. If l is an element of order 11 in $\langle a, b\rangle$, then $\bar{l}$ centralises unique involutions $\bar{u} \in \bar{U}$, $\bar{v} \in \bar{V}$ and $\bar{w} \in \bar{W}$. We will choose preimages u, v and w of $\bar{u}$, $\bar{v}$ and $\bar{w}$ in K such that:

- They generate the centraliser $2^2 \times \mathrm{D}_8$ of l,
- The automorphism σ of K which maps $U \to W \to V$ will map $u \to w \to v$.

As already described, the module for Co_1 restricts to K as

$$\mathbf{276a} \oplus \mathbf{22} \oplus \mathbf{276b} \oplus \mathbf{276c} \oplus \mathbf{48576} \oplus \mathbf{49152a} \oplus \mathbf{49152b} \oplus \mathbf{49152c}.$$

Here, **276a** and **22** come from the restriction of **298** to K. Restricting the tensor product $\mathbf{24} \otimes \mathbf{4096}$ gives

$$\mathbf{49152b} \oplus \mathbf{49152c} \cong \mathbf{24} \otimes (\mathbf{2048a} \oplus \mathbf{2048b}).$$

The remaining modules are summands of **98280**.

22 is a representation of the quotient group M_{24}, **276a**, **276b** and **276c** are representations of the three isomorphic quotient groups $2^{11}{:}\mathrm{M}_{24}$, **48576** is a representation of $K/Z(K) \cong 2^{11+11+11\cdot}\mathrm{M}_{24}$, and the three representations of degree 49152 are of the three isomorphic quotients $2^{1+11+11+11\cdot}\mathrm{M}_{24}$ obtained by factoring out each of the three non-trivial elements of the centre of K.

The module **98304** becomes monomial on restriction to K. But the image of K in **276b**$\oplus$**276c** has a monomial action on 2-dimensional subspaces, whereas on **98280** the group $2^{24}\mathrm{Co}_1$ acts monomially on 1-dimensional subspaces. We therefore cannot cut the pieces **276b** and **276c** from **98280** using only monomial-format programs. It is therefore worth converting the representation **552**, which is the direct sum of **276b** and **276c**, to ordinary matrix format. As all the modules of dimension 276 must be considered at once, all the smaller pieces can be "pasted" together and we can work with the representation **850**, where **850** = **276a** $\oplus$ **22** $\oplus$ **276b** $\oplus$ **276c**.

3.2 Words for some generators of K

As c and d are preimages for standard generators of Co_1, a good place to start looking for suitable words for (not necessarily standard) generators of K is in a list of words for maximal subgroups of Co_1 on standard generators. These are available in the WWW Atlas [9].

One such pair of words gives generators for the maximal subgroup of Co_1 of shape $2^{11}{:}\mathrm{M}_{24}$. The words are

$$e' = ((cdcd^2)^3)^{(cd)^6} \quad \text{and} \quad f' = ((((cd)^2cd^2)^2cd)^4)^{(cd^2)^5}.$$

Here, e' and f' generate a group which contains K but also contains an outer automorphism of K which acts by interchanging **49152b** with **49152c**, and **276b** with **276c**. This group is $N_G(K) \cong K.2$. Calculating in this group is relatively easy as it has a monomial action on **98304**.

We want to study the representations separately, so we must first put the 98280-dimensional matrices into block diagonal form. This was done by using the monomial standard basis program of [5]. The output of the standard basis program is our first basis change matrix in shorthand form, written as a monomial, which we will call $B_{\mathbf{98280}}$. After conjugating $e'_{\mathbf{98280}}$ and $f'_{\mathbf{98280}}$ by the inverse of this matrix, we can cut them into the pieces $e'_{\mathbf{552}}$, $e'_{\mathbf{48576}}$ and $e'_{\mathbf{49152a}}$, and $f'_{\mathbf{552}}$, $f'_{\mathbf{48576}}$ and $f'_{\mathbf{49152a}}$.

Next, words are found for the two generators a and b for a subgroup $2^{11\cdot}\mathrm{M}_{24}$ of K. We start looking for images of a and b near the top of the composition series. **22** contains insufficient information, as it represents only M_{24}, so we begin in **276a** which represents $2^{11}{:}\mathrm{M}_{24}$. Here we find the words

$$a''' = ((e'f')^4 e' f'^2 (e' f (e' f'^2)^3)^2)^8 \quad \text{and} \quad b''' = (e' f' e' f^2)^5$$

which generate a subgroup M_{24} in **276a**. On testing these words in **552** we find that they generate $2^{11}{:}\mathrm{M}_{24}.2$ modulo the subgroup 2^{2+11}. We can move into the subgroup of index 2 by taking the generators

$$a'' = (e'f')^{23} a''' \quad \text{and} \quad b'' = b'''.$$

We may obtain a complement to the subgroup of order 2^{11} by using the words

$$a' = ((a''b'')^5 b'' a'' b'' (a'' b''^2)^3 (a''b'')^2 b'')^6 \quad \text{and} \quad b' = ((a''b'')^4 b'' a'' b'' (a''b'')^3)^2,$$

which were found by a random search.

The generators are now almost as required, but their images in the representations of degree 49152 show that $\langle a', b' \rangle \cong 2^{1+11 \cdot}\mathrm{M}_{24} \cong 2 \times 2^{11 \cdot}\mathrm{M}_{24}$. Let y be the central involution. We find that $y = (a'b')^{23}$. We then have generators $a = a'y$ and $b = b'$ for the subgroup of $\langle a', b' \rangle$ of index 2.

4 Finding words for the remaining generators

4.1 Strategy

The other generators, u, v and w, can be found in a similar manner, but more care must be taken as it must be ascertained at each step that automorphisms of K exist which centralise a and b, and map u to w to v.

Recall that u, v and w are defined as generators for the centraliser in K of some element l, so we must first fix our element l. It must be an element whose image in the quotient group M_{24} has order 11. We choose $l = (bab)^2 (babbab^2)^2 (bab)^4 bab^2 bab (bab^2)^3$.

Note that u, v and w are inside K, so we first find generators $e = (e'f')^{23} e'$ and $f = f'$ for K itself.

4.2 Finding words for u, v and w modulo $Z(K)$

We can begin in the 850-dimensional representation which is the direct sum of **276a**, **22**, **276b** and **276c**. Here we find two words

$$u'' = ((efef^2)^8 l)^{11} \quad \text{and} \quad v'' = (((ef)^2(efef^2))^{15} l)^{11}$$

which yield two commuting involutions $u_{\mathbf{850}}$ and $v_{\mathbf{850}}$, and their product w'' yielding $w_{\mathbf{850}}$. Separately each of these extends $\langle a_{\mathbf{850}}, b_{\mathbf{850}} \rangle$ to 2^{11}:M$_{24}$ and together they extend it to $(2^{11} \times 2^{11})$:M$_{24}$.

We choose a basis with respect to which the five generators satisfy

$$a_{\mathbf{276a}} = a_{\mathbf{276b}} = a_{\mathbf{276c}},$$
$$b_{\mathbf{276a}} = b_{\mathbf{276b}} = b_{\mathbf{276c}},$$
$$u_{\mathbf{276a}} = v_{\mathbf{276a}} = u_{\mathbf{276b}} = w_{\mathbf{276b}} = v_{\mathbf{276c}} = w_{\mathbf{276c}}.$$

This is done using the MeatAxe program "zsb" (Standard Basis) which gives the basis change matrix $B_{\mathbf{850}}$.

In **48576** we can see $2^{11+22} {}^{\cdot} \mathrm{M}_{24} \cong K/Z(K)$, and $\langle a_{\mathbf{48576}}, b_{\mathbf{48576}} \rangle \cong 2^{11} {}^{\cdot} \mathrm{M}_{24}$. Both these groups have a normal subgroup Z of order 2^{11} which was not visible in **850**, and so our words for u, v and w are only correct modulo Z. Let u', v' and w' be words giving the correct images for u, v and w in **48576**, and let z be the involution centralising l in this newly visible 2-group. Clearly u' is either u'' or zu'', and similarly for v' and w'.

By looking at the orders of various words in the generators, we find that the correct words are

$$u' = zu'', \; v' = zv'' \; \text{ and } \; w' = w''.$$

4.3 Perfecting the words for u, v and w

The whole of K can be observed by looking in any two of the three representations of degree 49152 at once. This means that we can fully determine words for u, v and w by working in these representations.

Before we begin this section, the three generators are determined up to multiplication by elements of the centre $Z(K) \cong 2^2$. This group is generated by y and a second involution, $x = (e^2 f^2 eff^e f)^{23}$. In each of the 49152-dimensional representations, the image of one of the three non-trivial elements of $Z(K)$ is equal to the identity.

We can use the standard basis method again here to determine which, if any, of the elements of

$Z(K)$ must multiply each of u', v' and w' as we again know that **49152a**, **49152b** and **49152c** must be isomorphic to a fixed module **49152** when the respective generators $\{a, b, u, v\}$, $\{a, b, w, u\}$ and $\{a, b, v, w\}$ are used. First we must change basis on **98304** so that **49152b** and **49152c** can be worked in separately. This basis change can be saved as $B_{\mathbf{98304}}$, as it will be needed later.

We find that the three generating lists

$$\{a, b, u', yv'\},\ \{a, b, xw', u'\},\ \{a, b, yv', xw'\}$$

are isomorphic and the third and fourth generators in each case centralise l and together generate D_8.

The isomorphism test was performed by using a monomial standard basis program, based on the one used in [5]. The seed vector was the single coordinate vector fixed by a subgroup $2^{11}\mathrm{M}_{23}$, generated by

$$g = (((ab)^2ab^2ab)^4)^{(ab)^4} \quad \text{and} \quad h = (((ab)^2ab^2ab)^2)^{(ab^2)^3}.$$

The outputs when used on the respective generating sets were the standard basis matrices $B_{\mathbf{49152a}}$, $B_{\mathbf{49152b}}$ and $B_{\mathbf{49152c}}$.

The three new basis change matrices thus obtained can be recorded in a matrix

$$B_{\mathbf{147456}} = B_{\mathbf{49152a}} \oplus B_{\mathbf{49152b}} \oplus B_{\mathbf{49152c}}$$

5 One standard basis

5.1 One standard basis for 48576

The new generator T to be constructed in the next section is defined with respect to the standard basis obtained for the action of K on **196882**. So far, we have a standard basis for **850** and for **147456**, but we have not yet defined one for the remaining part of the module **48576**.

The situation is more complicated here than in the other two pieces of the module as we no longer have a unique monomial standard basis. In fact there are now three, and the action of σ on this part of the space can be described by a matrix which will map these bases to each other. In this section we will define one of them. The others will be defined in Section 6.2.

Before defining the basis, we must define generators for U, V and W. We can use the element ab of order 23 to define w_i to be $w^{(ab)^{i-1}}$, so $w_1 = w$ *etc.*, and define z_i, v_i and u_i similarly. We can see that $\langle z_1, \ldots, z_{11}\rangle = Z$ and $\langle w_1, \ldots, w_{11}\rangle = W$, and so the sets $\{u_1, \ldots, u_{11}\}$ and $\{v_1, \ldots, v_{11}\}$

must generate U and V respectively as these sets are the images of the first eleven w_i under the isomorphisms σ^2 and σ.

Let the products $v_{i_1} \dots v_{i_n}$ of n generators of V be denoted by $v_{i_1,\dots,i_n}$, and similarly products of generators of U, W and Z.

The first basis is defined by choosing its first element X as the centralised vector of

$$C_1 \cong \langle W, v_{2,4,8}, v_{2,3,5,6}, v_{3,5,9}, v_{7,3,2,1,11}, v_{7,5,4,2,1,10}, u_{2,4,8}, u_{2,3,5,6}, u_{3,5,9}, u_{7,3,2,1,11},$$
$$u_{7,5,4,2,1,10}, z_1, z_2, z_3, z_4, z_6, z_7, z_{5,8}, z_{5,9}, z_{5,10}, z_{5,11}\rangle$$

and the next 63 elements to be the images of X under the elements of $2^{11+11+11}$. In this group, $H_1 = \langle v_1, v_2, v_3, v_4, v_5, v_7\rangle$ is a complement to C_1, so we can find these vectors as the 63 images of X under the non-trivial elements of H_1.

The other words are chosen so that they will (up to signs) permute the 759 sets of 64 single coordinate vectors of which the 64 vectors just described form one set. This will give the full set of images of these 64-sets under the group $\langle a, b\rangle$, as it acts on these sets in a manner which corresponds to the action of M_{24} on the 759 octads of the Steiner system $S(5, 8, 24)$. Two such images can be found by taking the images of these vectors under the elements a and aba^2b, and the remaining sets can be found by using the elements k and j^2k defined below, of orders 11 and 23, with images in M_{24} that generate 23:11. The words used are given below in terms of the generators g and h for $2^{11}\mathrm{M}_{23}$, giving 253 images of these sets of 192 vectors.

The seed vector X is a single coordinate vector, so we can write $B_{\mathbf{48576}}$ using a monomial format standard basis program. We use one that gives the image of the seed vector under the words

$$v_1^{\alpha_1} v_2^{\alpha_2} v_3^{\alpha_3} v_4^{\alpha_4} v_5^{\alpha_5} v_7^{\alpha_6} a^{\beta} ((ab)^2 (abab^2)^2 ab^2)^{\gamma} (B)^{\delta} (A^2B)^{\epsilon},$$
$$0 \le \alpha_i \le 1 \text{ for each } \alpha,\ 0 \le \beta \le 1,\ 0 \le \gamma \le 1,\ 0 \le \delta \le 10,\ 0 \le \epsilon \le 22$$

where the words $\epsilon\delta\gamma\beta\alpha_1 \dots \alpha_6$ are in lexicographic order and

$$j = (h(ghgh^2)^2)^{((gh)^2 gh^2 (gh)^2 (ghgh^2)^2)^6)}$$

and

$$k = ((gh)^2 gh^2 (gh)^2 (ghgh^2)^2 gh^2)^{((gh)^2 gh^2 ghgh (ghgh^2)^2)^{-2})}.$$

5.2 The complete standard basis matrix

We now have three standard basis matrices $B_{\mathbf{850}}$, $B_{\mathbf{48576}}$ and $B_{\mathbf{147456}}$. We also have a monomial base change matrix $B_{\mathbf{98280}}$ which reorders the basis elements for the monomial representation of G into a more useful order.

We see that $B^{-1}_{\mathbf{98280}}$ conjugates the images in **98280** of generators for K to matrices of the form

$$\begin{pmatrix} *_{\mathbf{552}} & 0 & 0 \\ 0 & *_{\mathbf{48576}} & 0 \\ 0 & 0 & *_{\mathbf{49152a}} \end{pmatrix}$$

where $*$ denotes certain invertible matrices.

We can now write down the complete basis change matrix

$$B_{\mathbf{196882}} = \begin{pmatrix} B_{\mathbf{850}} & 0 & 0 \\ 0 & B_{\mathbf{48576}} & 0 \\ 0 & 0 & B_{\mathbf{147456}} \end{pmatrix} \begin{pmatrix} I_{850} & 0 & 0 \\ 0 & B_{\mathbf{98280}} & 0 \\ 0 & 0 & I_{98304} \end{pmatrix}.$$

Of course, $B_{\mathbf{196882}}$ cannot be written down in practice, as it is not monomial and therefore would not fit into our computers, but it will be useful when starting to calculate in the Monster to think of it in this way although it is actually stored in several different pieces.

6 Extending K to H

We know that T must act on the group K by mapping the generators a, b, u, v, w to a, b, w, u, v respectively, and we can now determine how it acts on the underlying vector space. It permutes the three summands **276a**, **276b** and **276c**, and **49152a**, **49152b** and **49152c**. We also know that it acts trivially on **22** and maps the different bases for **48576** to each other.

In this section, we construct a matrix with one of the possible actions on **196882**. The other cases will be obtained from it in Section 8.

6.1 The monomial part of an element T' extending K to H

When our generators of K are in standard basis the 850-dimensional matrices look like this:

$$a_{\mathbf{850}} = a_{\mathbf{276}} \oplus a_{\mathbf{22}} \oplus a_{\mathbf{276}} \oplus a_{\mathbf{276}}$$

$$b_{\mathbf{850}} = b_{\mathbf{276}} \oplus b_{\mathbf{22}} \oplus b_{\mathbf{276}} \oplus b_{\mathbf{276}}$$

$$u_{\mathbf{850}} = M \oplus I_{\mathbf{22}} \oplus M \oplus I_{\mathbf{276}}$$

$$v_{\mathbf{850}} = M \oplus I_{\mathbf{22}} \oplus I_{\mathbf{276}} \oplus M$$

$$w_{\mathbf{850}} = I_{\mathbf{276}} \oplus I_{\mathbf{22}} \oplus M \oplus M$$

where $M = u_{\mathbf{276a}}$ $(= u_{\mathbf{276b}} = v_{\mathbf{276a}} = w_{\mathbf{276c}} = w_{\mathbf{276b}} = w_{\mathbf{276c}})$ is some 276-dimensional matrix of order 2.

$T'_{\mathbf{850}}$ can be described as a matrix which will conjugate the four elements $a_{\mathbf{850}}, b_{\mathbf{850}}, u_{\mathbf{850}}, v_{\mathbf{850}}$ to $a_{\mathbf{850}}, b_{\mathbf{850}}, w_{\mathbf{850}}, u_{\mathbf{850}}$. Such a matrix could be produced by using standard basis programs, but in this case it is obvious that one such matrix is

$$\begin{pmatrix} 0 & 0 & I_{276} & 0 \\ 0 & I_{22} & 0 & 0 \\ 0 & 0 & 0 & I_{276} \\ I_{276} & 0 & 0 & 0 \end{pmatrix}.$$

Moreover, we know that $T_{\mathbf{850}}$ acts trivially on **22** as this is a representation of M_{24}, and the only elements of K with non-trivial images in this representation are contained in $\langle a, b \rangle$, the subgroup centralised by σ. (Here we use the fact that T has order 3, but by Schur's Lemma σ, and hence T, acts as a scalar ± 1 on **22**.)

The matrix $T'_{\mathbf{147456}}$ will look very similar to $T'_{\mathbf{850}}$. Here the standard generators for K act as

$$a_{\mathbf{147456}} = a_{\mathbf{49152}} \oplus a_{\mathbf{49152}} \oplus a_{\mathbf{49152}}$$

$$b_{\mathbf{147456}} = b_{\mathbf{49152}} \oplus b_{\mathbf{49152}} \oplus b_{\mathbf{49152}}$$

$$u_{\mathbf{147456}} = M \oplus N \oplus R$$

$$v_{\mathbf{147456}} = N \oplus R \oplus M$$

$$w_{\mathbf{147456}} = R \oplus M \oplus N$$

with respect to the standard basis, where M, N and R are three particular 49152-dimensional matrices of order 2.

Again, we can write down one suitable matrix

$$T'_{\mathbf{147456}} = \begin{pmatrix} 0 & I_{49152} & 0 \\ 0 & 0 & I_{49152} \\ I_{49152} & 0 & 0 \end{pmatrix}.$$

6.2 The non-monomial part of T'

Unlike the other direct summands of T', this part is not in a format that we already have programs to calculate with. This is because it must conjugate images for the generators of K in essentially different monomial representations to each other. Indeed, it must conjugate the diagonal matrices representing W to proper monomials representing V.

In Section 5 we gave the first standard basis as the set of images of a seed vector under a set of 48576 words in a, b, u, v and w. The other two standard bases can be obtained by replacing u, v, w with w, u, v and v, w, u in these words. The three bases will be distinct as the words are not symmetric in these three generators. As we know that σ acts by mapping $u \to w \to v$ and centralising a and b it is obvious that it will act as required on these three bases.

Recall that the first element of the first basis was defined to be the centralised vector of

$$C_1 = \langle W, v_{2,4,8}, v_{2,3,5,6}, v_{3,5,9}, v_{7,3,2,1,11}, v_{7,5,4,2,1,10},$$
$$u_{2,4,8}, u_{2,3,5,6}, u_{3,5,9}, u_{7,3,2,1,11}, u_{7,5,4,2,1,10}, z_1, z_2, z_3, z_4, z_6, z_7, z_{5,8}, z_{5,9}, z_{5,10}, z_{5,11}\rangle.$$

As T' must fix z and conjugate w to v, u to w and v to u, the first element of the second basis must be the vector centralised by

$$C_2 = \langle V, u_{2,4,8}, u_{2,3,5,6}, u_{3,5,9}, u_{7,3,2,1,11}, u_{7,5,4,2,1,10},$$
$$w_{2,4,8}, w_{2,3,5,6}, w_{3,5,9}, w_{7,3,2,1,11}, w_{7,5,4,2,1,10}, z_1, z_2, z_3, z_4, z_6, z_7, z_{5,8}, z_{5,9}, z_{5,10}, z_{5,11}\rangle.$$

When the generators for K are written with respect to the first basis the first coordinate vector

$$X_1 = (1\,0\,0\ \ldots\ \ldots\ 0)$$

is centralised by C_1. Thus the first row of the matrix which changes this basis to the second one is a centralised vector X_2 of C_2.

As the first 64 elements of the first standard basis were defined in Section 5 to be the images of $\pm X_1$ under the elements of $H_1 = \langle v_1, v_2, v_3, v_4, v_5, v_7 \rangle$, the first row of the matrix will be

$$\begin{array}{rcc} X_2 = \pm & (1 \; \ldots \; 1 & 0 \; \ldots \ldots \ldots \; 0) \\ & \leftarrow 64 \rightarrow & \leftarrow 48512 \rightarrow \end{array}$$

The next 63 rows of the matrix are given by taking the images of this vector under $H_2 = \langle u_1, u_2, u_3, u_4, u_5, u_7 \rangle$ in the same order we used to give the first 64 rows of the first basis.

With respect to the first standard basis, each element of U is the product of an element of V and a diagonal element in W. So, while the first 64 entries of the image of X_2 under these elements may have different signs, they will all be non-zero and the final 48512 coordinates will remain as zeroes.

This gives the first 64 rows of the matrix as:

$$\begin{array}{c} \uparrow \\ 64 \\ \downarrow \end{array} \left(\begin{array}{ccc} A & \vdots & 0 \\ \leftarrow 64 \rightarrow & & \leftarrow 48512 \rightarrow \end{array} \right)$$

where A is the non-zero part of this matrix.

Recall from section 5.1 that the set of 64 images of X_1 under $2^{11+11+11}$ had 759 images under certain words in a and b, and that a and b acted (modulo signs) as permutations of these 759 sets of vectors. With respect to the first standard basis, the words preserved the ordering of the vectors in each set, but shifted the non-zero coordinates a multiple of 64 places to the right. Thus taking the images of the matrix shown above under these same words will give us the remaining rows of the matrix

$$\begin{pmatrix} A & & 0 \\ & \ddots & \\ 0 & & A \end{pmatrix}$$

in which A is repeated 759 times down the diagonal.

We know that T_{48576} will have order 3. This allows us to determine the sign of X_2, the first row of the matrix. If we choose "+", the order of the matrix is 6, but "−" gives order 3 as required.

In practice, there is no need to store the whole matrix as all the information required is contained in the 64×64 matrix A.

7 Calculating in $\langle G, T\rangle$

All the candidates for T can be obtained from T' by multiplication by certain diagonal matrices which will be determined in the next section. This means that any methods which can be used to calculate in $\langle c, d, T'\rangle$ will work for any possibility for T.

The only main operation that can feasibly be performed in $\langle c, d, T'\rangle$ is that of finding a factor of the order of elements. The matrices are too large to use standard programs, but we can find a factor of the order of a word $\alpha(c, d, T')$ by calculating the size of an orbit of $\alpha(c, d, T')$ on a 196882-dimensional vector. Any sufficiently random vector will suffice. If the word contains two adjacent letters which are elements of G, these can be multiplied together in the usual way. The only pairs of elements which cannot be multiplied together are pairs consisting of T' (or its inverse) and one element of G. As the image of a vector under a word can be found by multiplying the vector once by each of the elements in the word in order, the question of how to multiply a vector by a word reduces to one of how to multiply it by a product $C_1T'^{\varepsilon_1}C_2T'^{\varepsilon_2}...$ for some elements $C_1, C_2, ... \in G$.

7.1 Inventory

Before describing the method used to multiply vectors by matrices, it is worth remembering how our generating matrices are stored.

At this stage we have the following files. For each of the two generators of G we have

- a 298×298 matrix
- a 24×24 matrix
- a 4096×4096 matrix
- a 98280×98280 matrix stored in monomial format.

and for T' and its inverse we have

- an 850×850 matrix
- a 147456×147456 matrix in monomial format
- a 64×64 matrix.

We also have some base change matrices:

- a 98280×98280 monomial format matrix

- a 48576×48576 monomial format matrix
- a 147456×147456 monomial format matrix
- an 850×850 matrix.

Throughout the rest of the section, let C be some element of G.

7.2 An outline of the method

The method used for multiplying a vector X by CT' or CT'^{-1} is as follows:

1. Cut X into the three pieces $X_{\mathbf{298}}$, $X_{\mathbf{98280}}$ and $X_{\mathbf{98304}}$.
2. Multiply each vector $X_{\mathbf{n}}$ by the corresponding matrix $C_{\mathbf{n}}$.
3. Multiply X by the inverse of the base change matrices in order, cutting, pasting and changing format where necessary.
4. Calculate XT' or XT'^{-1} as appropriate.
5. Change basis on the vector again, this time multiplying it by the base change matrices themselves.

This process can be repeated for each such pair of adjacent factors in the word $\alpha(c, d, T')$.

After T has been fully determined, it will be possible to shorten the process by writing all three generators with respect to a "compromise" basis. We can conjugate $T_{\mathbf{147456}}$ and $T_{\mathbf{850}}$ by $B_{\mathbf{147456}}$ and $B_{\mathbf{850}}$ respectively, and $c_{\mathbf{98280}}$ and $d_{\mathbf{98280}}$ by $B^{-1}_{\mathbf{98280}}$.

7.3 The details of the method

Multiplication of $X_{\mathbf{850}}$ by $T'_{\mathbf{850}}$ and of $X_{\mathbf{298}}$ by $C_{\mathbf{298}}$ is trivial, and so is calculating the vector–monomial products $X_{\mathbf{98280}}C_{\mathbf{98280}}$ and $X_{\mathbf{147456}}T'_{\mathbf{147456}}$, but we need another method for the other parts of the matrices.

As the 48576-dimensional matrix for T' has 759 identical matrices A on the diagonal and zeroes everywhere else, we have only stored A. Also, using standard multiplication programs on such large matrices would not have been an option. But we can multiply vectors by $T'_{\mathbf{48576}}$ by "folding" $X_{\mathbf{48576}}$ into a 759×64 matrix, postmultiplying this by A, then "unfolding" the result.

There is a similar method for multiplying vectors by the images of elements of $2^{1+24}_{+}{}^{\cdot}\mathrm{Co}_1$ in the 98304-dimensional tensor product space. A standard method would not be practical as our computers are unable to deal with 98304-dimensional matrices. Instead, we multiply the vector $X_{\mathbf{98304}}$ by the tensor product $C_{\mathbf{4096}} \otimes C_{\mathbf{24}}$ by folding $X_{\mathbf{98304}}$ into X', a 24 × 4096 matrix, calculating $C^t_{\mathbf{24}} X' C_{\mathbf{4096}}$ and then unfolding the result.

Multiplying vectors by words is a lot slower in this construction than in the $GF(2)$ construction of [5]. It takes approximatesly six seconds to multiply a vector by the word cT' using a Pentium II/450MHz processor with 384 MB of RAM, while a similar computation in the other construction takes about a hundredth of this time.

8 Generating the Monster

8.1 What are the cases?

The different cases for $\langle c, d, T\rangle$ arise because T is not uniquely determined by the action of σ on K. This section looks at the various possibilities.

It is obvious that the action of T' on K is fixed by conjugation of T' by any matrix centralising the generators of K. This means that we can conjugate T' by any element of the elementary abelian group of order 2^8 generated by elements acting as -1 on any one of the summands of the module for K, and $+1$ on the other seven summands.

The only elements of this group which can possibly act non-trivially on T' are the elements of the subgroup of order 2^6 generated by elements acting as -1 on any one of **276a**, **276b**, **276c**, **49152a**, **49152b** or **49152c**. This group then contains a subgroup of order 4, generated by the element acting as -1 on **276a**, **276b** and **276c** and the one acting as -1 on $\mathbf{147456} = \mathbf{49152a} + \mathbf{49152b} + \mathbf{49152c}$, which will also centralise T'. Modulo this subgroup, we are left with 2^4 possibilities to consider.

Now suppose we have chosen to conjugate T' by some element κ centralising c and d. This will not give us an essentially different group as

$$\langle c, d, T^\kappa\rangle \cong \langle c^\kappa, d^\kappa, T^\kappa\rangle \cong \langle c, d, T\rangle^\kappa \cong \langle c, d, T\rangle.$$

We now consider what the different choices for T are, and eliminate those which do not give a different group.

Consider the two matrices

$$-I_{\mathbf{276}} \oplus I_{\mathbf{22}} \oplus I_{\mathbf{276}} \oplus I_{\mathbf{276}} \quad \text{and} \quad I_{\mathbf{276}} \oplus I_{\mathbf{22}} \oplus -I_{\mathbf{276}} \oplus I_{\mathbf{276}}$$

Conjugation of $T'_{\mathbf{850}}$ by either of these matrices or by their product will not affect the action on K as they centralise the image of K itself in this representation. So conjugation by any one of them gives a viable alternative to $T'_{\mathbf{850}}$.

Four possibilities also arise for $T_{\mathbf{147456}}$ in a similar manner. These are $T'_{\mathbf{147456}}$ and conjugation of it by any one of the three matrices ω, χ and ψ which act as -1 on **49152a**, **49152b**, and **49152c** repectively and as the identity elsewhere.

As any candidate for $T_{\mathbf{147456}}$ can be chosen independently of the one for $T_{\mathbf{850}}$, this gives us a total of $4 \times 4 = 16$ cases to consider, as stated above.

8.2 Which of the cases give different results?

Let us first consider $T_{\mathbf{147456}}$.

Modulo the element $-I_{\mathbf{147456}}$ which centralises both T' and the generators for K, the cases described above correspond to conjugation of $T'_{\mathbf{147456}}$ by the matrices

$$I_{\mathbf{49152a}} \oplus -I_{\mathbf{49152b}} \oplus -I_{\mathbf{49152c}}, \quad -I_{\mathbf{49152a}} \oplus I_{\mathbf{49152b}} \oplus -I_{\mathbf{49152c}}$$

and

$$-I_{\mathbf{49152a}} \oplus -I_{\mathbf{49152b}} \oplus I_{\mathbf{49152c}}.$$

Note that these are respectively the involutions $(xy)_{\mathbf{147456}}$, $x_{\mathbf{147456}}$ and $y_{\mathbf{147456}}$, and that the image of these involutions is the identity in all the other represenations. So conjugating $T'_{\mathbf{147456}}$ by one of these three matrices has the same effect as conjugating T' by the corresponding element of $Z(K)$.

It is obvious that the group $\langle c, d, T' \rangle$ must be equal to $\langle c, d, T'^{\alpha(c,d)} \rangle$ as for any word α

$$(T')^{\alpha(c,d)} \in \langle c, d, T' \rangle$$

and

$$T' = ((T')^{\alpha(c,d)})^{(\alpha(c,d)^{-1})} \in \langle c, d, (T')^{\alpha(c,d)} \rangle$$

and therefore all four possibilities for T obtained by choosing $T_{\mathbf{147456}}$ differently will extend $\langle c, d \rangle$ to the same group.

This argument does not apply in **850** as no central elements of K are visible, but we can still show that there are only two different cases arising here.

Let κ be the matrix

$$-I_{\mathbf{276}} \oplus I_{\mathbf{22}} \oplus I_{\mathbf{276}} \oplus I_{\mathbf{276}} \oplus I_{\mathbf{98280}} \oplus I_{\mathbf{98304}}$$

It is shown below that $T'^{\kappa} = T'^{\tau}$ for some matrix τ which centralises not only K but the whole of G, and it has already been noted that conjugation of T' by any such matrix will not afford anything new.

The matrix κ acts uniquely on T' up to multiplication by any element centralising T'. For example, replacing the submatrix $I_{\mathbf{22}}$ by $-I_{\mathbf{22}}$ will not affect this action. This gives the matrix τ which acts as -1 on the 298-dimensional constituent of the module for G and $+1$ elsewhere. It will thus centralise any element of G.

The third and fourth choices of $T_{\mathbf{850}}$ act in the same way modulo κ, and so now we can see that they both are in effect the same case. This leaves us with a total of only two cases to consider instead of the original 16.

8.3 Determining which is the case that we want

We must now differentiate between the two cases and decide which one is our third Monster generator.

This is done computationally, using the method given in Section 8 to find factors of element orders. This is sufficient to distinguish between the cases as we know that the largest element order in the Monster is 119, whereas in most subgroups of $GL_{196882}(3)$ it is a much greater figure. So if some element can be found in one of the cases with order greater than 119, this case cannot be the Monster.

We find that $T = T'$, as by using this choice we find the words cT, cdT and cd^2T to have orders 60, 87 and 46 respectively, but in the second case no non-trivial words were found with order less than 120.

9 Conclusions

We have now constructed three matrices, the first two of which generate an involution centraliser in $\mathbb{M}$, and the third which extends this group to the Monster itself. We have also found a practical method of calculating within our new representation.

One application of the construction is to help to determine the maximal subgroups of $\mathbb{M}$. It has so far been used to classify the subgroups of $\mathbb{M}$ isomorphic to $\mathrm{L}_2(23)$, to find a subgroup $\mathrm{L}_2(29)$ contained in the new maximal subgroup $\mathrm{L}_2(29)$:2, and also to show that there are no subgroups isomorphic to $\mathrm{L}_2(13)$ which contain elements of class $13B$ (see [4]). We hope to extend this work to classify several other isomorphism types of possible maximal subgroups.

10 Index of Notation

Here is a list of the notation which is used frequently throughout, together with brief definitions. All modules are over $GF(3)$ unless stated otherwise.

22	a module for M_{24}
24	a module for $2\mathrm{Co}_1$
24f2	a module for Co_1 over GF(2)
276a,b,c	isomorphic modules for three different quotients $2^{11}{:}\mathrm{M}_{24}$ of K
298	a module for Co_1
850	a module for $2^{22}{:}(\mathrm{M}_{24} \times 3)$
4096	a module for a group $2_+^{1+24}\mathrm{Co}_1$ not isomorphic to G
48576	a module for $2^{11+22\cdot}(\mathrm{M}_{24} \times 3)$
49152a,b,c	isomorphic monomial modules for three different quotients $2^{1+11+22\cdot}\mathrm{M}_{24}$ of K
98280	a monomial representation of $2^{24\cdot}\mathrm{Co}_1$
98304	**24**$\otimes$**4096**
147456	a monomial representation of $2^{2+11+22\cdot}(\mathrm{M}_{24} \times 3)$
196882	the module for **M**
a, b	generators for $C_G(T) \cong 2^{11\cdot}\mathrm{M}_{24}$
B	a basis change matrix
c, d	generators for G
E	$O_2(G) \cong 2_+^{1+24}$
G	$C_{\mathbb{M}}(z) \cong 2_+^{1+24\cdot}\mathrm{Co}_1$
H	$\langle T, K\rangle$
$g_1, \ldots, g_{24}$	generators for E
K	$C_G(y) \cong 2^{2+11+22\cdot}\mathrm{M}_{24}$, normalised by T
P	permutation representation of Co_1 on 98280 points
T	the third generator of $\mathbb{M}$
T'	a candidate for the third generator of $\mathbb{M}$
u, v, w	involutions in E extending $\langle a, b\rangle$ to K
U, V, W, Z	subgroups of $O_2(K)$ containing u, v, w, z respectively
x	an element in class $2B$ centralized by G
y	an element of E in class $2B$
z	an involution in $O_2(\langle a, b\rangle)$
Λ	an orbit of 98280 vectors in **24f2**

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