Finite simple groups **Problem sheet 3**

EXERCISE 1. *Show that if f is any bilinear or sesquilinear form on a vector space V, and $S^\perp = \{v \in V \mid f(u,v) = 0$ for all $u \in S\}$, then $S^\perp$ is a subspace of V.*

EXERCISE 2. *Show that if f is a non-singular bilinear or sesquilinear form, and U is a subspace of V, then $(U^\perp)^\perp = U$ and $\dim(U)+\dim(U^\perp) = \dim(V)$. Deduce that if $U \cap U^\perp = 0$ then $V = U \oplus U^\perp$.*

EXERCISE 3. *Let f be a non-singular alternating form on a vector space V of dimension $2m$ over $\mathbb{F}_q$. If $k \leq m$, how many non-singular subspaces of dimension $2k$ are there in V? How many totally isotropic subspaces of dimension k are there?*

EXERCISE 4. *Show that the symplectic transvections $T_v(\lambda) : x \mapsto x + \lambda f(x,v)v$ preserve the alternating bilinear form f.*

EXERCISE 5. *Verify that the symplectic transvections are commutators in $\mathrm{Sp}_4(3)$ and $\mathrm{Sp}_6(2)$.*

EXERCISE 6. *Show that the unitary transvections $T_v(\lambda) : x \mapsto x + \lambda f(x,v)v$ preserve the non-singular conjugate-symmetric sesquilinear form f if and only if $\lambda^{q-1} = -1$.*

EXERCISE 7. *Let V be a 3-dimensional space over $\mathbb{F}_4 = \{0, 1, \omega, \omega^2\}$, and let f be a non-singular conjugate-symmetric sesquilinear form on V. Show that there are 21 one-dimensional subspaces of V, of which 9 contain isotropic vectors and 12 contain non-isotropic vectors.*

EXERCISE 8. *From the previous question we get an action of $\mathrm{GU}_3(2)$ (and also of $\mathrm{PGU}_3(2)$) on the set of nine isotropic 1-spaces in V. Show that this action is 2-transitive, and that $|\mathrm{PGU}_3(2)| = 216$.*

Deduce (from the O'Nan–Scott theorem, or otherwise) that the resulting subgroup of A_9 is the 'affine' subgroup $(C_3 \times C_3){:}\mathrm{SL}_2(3)$.

EXERCISE 9. *Show that the 2-dimensional orthogonal groups are dihedral; specifically: $\mathrm{O}_2^+(q) \cong D_{2(q-1)}$ and $\mathrm{O}_2^-(q) \cong D_{2(q+1)}$, both for q odd and q even.*