Hints and solutions to selected exercises

Exercise 1. *For a permutation* $\pi \in S_n$ *define*

$$\varepsilon(\pi) = \prod_{1 \le i < j \le n} \frac{i-j}{i^\pi - j^\pi} \in \mathbb{Q}.$$

Show that $\varepsilon = \pm 1$ *and that* ε *is a group homomorphism from* S_n *onto* $C_2 = \{1, -1\}$*. Hence obtain another proof that the sign of a permutation is well-defined.*

Exercise 2. *Let* $G < S_n$ *act transitively on* $\Omega = \{1, \ldots, n\}$ *and let* $H = \{g \in G : a^g = a\}$ *for fixed* $a \in \Omega$*. Prove that* $\phi : a^g \mapsto Hg$ *is a bijection between* Ω *and the set* $G : H$ *of right cosets of* H *in* G*.*

Prove also that $Hg = \{x \in G : a^x = a^g\}$*.*

If $a^g = a^k$ then $a^{gk^{-1}} = a$ so $gk^{-1} \in H$ so $g \in Hk$, whence $Hg = Hk$, so ϕ is well-defined. The same argument in reverse shows ϕ is one-to-one. Clearly ϕ is onto, so ϕ is a bijection.

The second part is essentially the same: if $x \in Hg$ then $x = hg$ for some $h \in H$, so $a^x = a^{hg} = a^g$ and conversely.

Exercise 3. *Prove that the orbits of a group* H *acting on a set* Ω *form a partition of* Ω*.*

Exercise 4. *Show that* A_n *is not* $(n-1)$*-transitive.*

If the $n-1$ points $1, 2, 3, \ldots, n-2, n-1$ are mapped to $1, 2, 3, \ldots, n-2, n$ respectively, then n is mapped to $n-1$, and the permutation is $(n-1, n)$, which is not in A_n.

Exercise 5. *Let* G *act transitively on* Ω*. Show that the average number of fixed points of the elements of* G *is* 1*, i.e.*

$$\frac{1}{|G|} \sum_{g \in G} |\{x \in \Omega \mid x^g = x\}| = 1.$$

Count the pairs $(x, g) \in \Omega \times G$ such that $x^g = x$ in two ways. In one way it is $\sum_{g \in G} |\{x \in \Omega \mid x^g = x\}|$. In the other way it is (where H is a point stabilizer) $\sum_{x \in \Omega} |\{g \in G \mid x^g = x\}| = |\Omega|.|G : H| = |G|$ by the orbit–stabilizer theorem.

Exercise 6. *Verify that the semidirect product* $G :_\phi H$ *is a group. Show that the subset* $\{(g, 1_H) : g \in G\}$ *is a normal subgroup isomorphic to* G*, and that the subset* $\{(1_G, h) : h \in H\}$ *is a subgroup isomorphic to* H*.*

(Sorry, part of the question was missing.)

We need to check the identity law, the inverse law and the associative law. The associative law is the hardest: on the one hand

$$\begin{aligned}((g_1,h_1)(g_2,h_2))(g_3,h_3) &= (g_1 g_2^{\phi(h_1^{-1})}, h_1h_2)(g_3,h_3)\\ &= ((g_1 g_2^{\phi(h_1^{-1})}) g_3^{\phi(h_1h_2)^{-1}}, h_1h_2h_3)\end{aligned}$$

and on the other

$$\begin{aligned}(g_1,h_1)((g_2,h_2)(g_3,h_3)) &= (g_1,h_1)(g_2 g_3^{\phi(h_2^{-1})}, h_2h_3)\\ &= (g_1(g_2 g_3^{\phi(h_2^{-1})})^{\phi(h_1^{-1})}, h_1h_2h_3)\\ &= (g_1 g_2^{\phi(h_1^{-1})} g_3^{\phi(h_2^{-1})\phi(h_1^{-1})}, h_1h_2h_3)\end{aligned}$$

which is the same thing since ϕ is group homomorphism.

The map $(g,1_H)\mapsto g$ is an isomorphism because $(g,1_H)(k,1_H) = (gk,1_H)$ and similarly $(1_G,h)\mapsto h$ is an isomorphism because $(1_G,h)(1_G,l) = (1_G,hl)$. In particular the given sets are groups. The first is normal because

$$(1_G,h^{-1})(g,1_H)(1_G,h) = (1_G,h^{-1})(g,h) = (g^{\phi(h)},1_H).$$

Exercise 7. *Suppose that G has a normal subgroup A and a subgroup B satisfying $G = AB$ and $A\cap B = 1$. Prove that $G \cong A{:}_\phi B$, where $\phi : B \to \mathrm{Aut}A$ is defined by $\phi(b) : a \mapsto b^{-1}ab$.*

Look it up in a suitable textbook.

Exercise 8. *Prove that if the permutation π on n points is the product of k disjoint cycles (including trivial cycles), then π is an even permutation if and only if $n-k$ is an even integer.*

If $k = n$ then $\pi = 1$ and the result is true. Now use induction on $n-k$. To reduce the number of cycles by 1, multiply by a transposition that straddles two cycles: thus $(a_1,a_2,\ldots,a_r)(b_1,b_2,\ldots,b_s)(a_1,b_1) = (a_1,a_2,\ldots,a_r,b_1,b_2,\ldots,b_s)$. Thus each time the number of cycles decreases by 1, the permutation changes from even to odd or vice versa.

Exercise 9. *Determine the number of conjugacy classes in A_8, and write down one element from each class.*

The cycle types of even permutations are 71, 62, 53, 5111, 44, 4211, 3311, 311111, 3221, 221111 and 11111111. Of these, just 71 and 53 consist of disjoint cycles of distinct odd lengths, so split into two classes in A_8. In particular, there are exactly 13 conjugacy classes in A_8. Thus $(1,2,3,4,5,6,7,8)$ and $(1,2,3,4,5,6,8,7)$ are in different classes, as are $(1,2,3,4,5)(6,7,8)$ and $(1,2,3,4,5)(6,8,7)$. In the other cases, any element will do, e.g. $(1,2,3,4,5,6)(7,8)$, $(1,2,3,4,5)$, $(1,2,3,4)(5,6,7,8)$, etc.