Queen Mary and Westfield College
University of London
BSc Examination by Course Units 1998

MAS 205 Complex Variables

10.00 am Friday 29th May 1998

The duration of this examination is 2 hours.

This paper has two Sections and you should attempt both Sections. Please read carefully the instructions given at the beginning of each Section.

Calculators may be used in this examination, but any programming, graph plotting or algebraic facility may not be used. Please state on your answer book the name and type of machine used.

SECTION A *Each question carries 12 marks. You should attempt ALL questions.*

A1.
(i) Find all solutions of the equation

$$z^3 + 8i = 0$$

(ii) Let $z = x + iy$ and $w = u + iv$. Consider the transformation

$$w = \frac{1}{z+1}$$

mapping the z-plane to the w-plane. Show that the imaginary axis $(x = 0)$ in the z-plane maps to the circle

$$u^2 + v^2 - u = 0$$

in the w-plane. What is the image of the set $\{z : Re(z) > 0\}$?

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A2.
(i) Find the value of each of the following limits:

$$\lim_{z\to\infty} \frac{2z^2-1}{z^2+1} \qquad\qquad \lim_{z\to 0} \frac{z}{\tan z}$$

(ii) Let $f=u+iv$ be a complex function of a comples variable $z=x+iy$. Write down the *Cauchy-Riemann equations* satisfied by u and v at points where f is complex differentiable. Find the real and imaginary parts u and v of the function $f(z)=e^{iz}$ and verify that they satisfy the Cauchy-Riemann equations for all values of z.

A3.
(i) Let

$$f(z) = \frac{z}{1-2z}$$

Find the Taylor series for f about the point $z=0$. What is the radius of convergence of this series ?

(ii) Let

$$g(z) = \frac{e^z}{z-1}$$

Find the Laurent series for g about the singularity $z=1$. What is the value of the residue of g at this point ?

A4.
Define what is meant by the *integral* of a complex function f along a piecewise smooth curve C parametrised by a path γ.

Evaluate

$$\int_C f(z)dz$$

where $f(z)=z^2+(\bar{z})^2$ and C is the straight line path along the imaginary axis from $z=-i$ to $z=+i$.

A5.
Evaluate

$$\int_C \frac{1}{z(z^2+4)}dz$$

where C is the positively oriented circle with centre $z=1$ and radius r, in each of the two following cases:

(i) $r=1/2$;
(ii) $r=2$.

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SECTION B *Each question reties 20 marks. You may attempt all questions but only marks for the best TWO questions will be counted.*

B6.

What are meant by the statements that a function $f : \mathrm{C} \to \mathbf{C}$ is *differentiable* at a point $z_0 \in \mathbf{C}$, and that f is *homomorphic* on an open set U c C ? Prove that if f is differentiable at z_0 then the Cauchy-Riemann equations are satisfied there. Deduce that if f is homomorphic on an open set U then the real and imaginary parts u and v of f are *harmonic* on U.

Let $u : \mathbf{R}^2 \to \mathbf{R}$ be the function

$$u(x, y) = x(2y + 1)$$

Show that u is harmonic, find a harmonic conjugate v for u, and hence or otherwise find a homomorphic function $f : \mathrm{C} \to \mathrm{C}$ such that u is the real part of f.

B7.

State *Cauchy's Theorem* and explain briefly why it follows from this theorem that if f is homomorphic on the region between two simple closed contours Cl and C_2, where C_2 is in the interior of the region bounded by Cl, then

$$\int_{C_1} f(z)dz = \int_{C_2} f(z)dz$$

Let ϕ be a function homomorphic on and everywhere inside a simple closed contour C. State and prove *Cauchy's Integral Formula* for the value of ϕ at any point z_0 inside C. *(You may assume that for a small circle C' having z_0 as centre $\int_{C'} \frac{1}{z - z_0} dz = 2\pi i$.)* Using Cauchy's Integral Formula, or otherwise, compute the value of

$$\int_C \frac{e^z}{z + 1}$$

where C is the (positively oriented) circle having centre $z =$ O and radius 2.

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B8.

(i) *Cauchy's Inequality* states that if f is homomorphic on and inside a title C with centre z_0 and radius R then

$$|f'(z_0)| \leq \frac{M_R}{R}$$

where M_R is the maximum value of $|f(z)|$ among points z lying on the circle C. Assuming Cauchy's Inequality prove *Liouville's Theorem* that any entire bounded function is constant, and hence or otherwise prove the *fundamental Theorem of Algebra.*

(ii) State *Rouché's Theorem.* How many zeros (counted with multiplicity) does the polynomial

$$p(z) = z^4 - 5z^2 + 8z - 1$$

have in the annulus $\{z : 1 < |z| < 3\}$?.

B9.

(i) Define what is meant by an *isolated singularity* of a complex function f, define the three *types* of isolated singularity, and define the *residue* of f at such a singularity.

Let

$$f(z) = \frac{\sin(\pi z)}{z(2z-1)}$$

What are the types of the singularities of f at $z = 0$ and at $z = 1/2$?

(ii) Evaluate the real integral

$$\int_{-\infty}^{\infty} \frac{1}{(x^2+1)(x^2-2x+2)} dx$$

End of examination paper