Queen Mary and Westfield College
University of London
BSc Examination 1999

MAS205 Complex Variables

Friday 4th June 1999 10.00am

The duration, of this examination is 2 hours.

This paper has two Sections and you should attempt both Sections. Please read carefully the instructions given at the beginning of each Section.

Calculators may be used in this examination, but any programming, graph plotting or algebraic facility may not be used. Please state on your answer book the name and type of machine used.

SECTION A *Each question carries 12 marks. You should attempt ALL questions.*

A1.
(i) Find all solutions of the equation

$$e^{4z} = 1.$$

(ii) Evaluate

$$(a) \quad \lim_{z\to i}\left(\frac{z^2-2iz-1}{z^3-iz^2+z-i}\right) \qquad (b) \quad \lim_{z\to\infty}\left(\frac{1+z+z^2}{i+z-iz^2}\right).$$

Az.
Find the Mobius transformation α:

$$z \to w = \frac{az+b}{cz+d}$$

which has $\alpha(0) = 1, \alpha(1) = -i$ and $\alpha(-1) = i$.
Under this transformation, what is the image in the w-plane of the real axis in the z-plane?
What is the image in the w-plane of the upper half of the z-plane?

Next question on next page

A3.

Define what is meant by saying that a complex function f is *differentiable* at $z \in \mathbb{C}$.

For what values of z is each of the following functions differentiable?

(i) $f(z) = z\overline{z}$ (where $\overline{z}$ is the complex conjugate of z);

(ii) $f(z) = (x^2 + y^2) + i(x^2 - y^2)$ (where $z = x + iy$).

A4.

(i) Let

$$f(z) = \frac{z - 1}{z}.$$

Find the Taylor series $\sum_0^\infty a_n (z - 1)^n$ for f about the point $z = 1$.

For what values of z does this series converge absolutely'?

(ii) Let

$$g(z) = \frac{1}{22(4\_ \quad *2)''}$$

Find the Laurent series $\sum_0^\infty a_n z^n + \sum_1^\infty b_n z^{-n}$ for g about the point $z = 0$.

What kind of singularity does g have at z = O ? What is the residue of g there?

A5.

Evaluate

$$\int_C \frac{\sin(\pi z)}{2z - 3} dz$$

in each of the following two cases:

(i) C is the positively oriented circle with centre z = O and radius 1;

(ii) C is the positively oriented circle with centre $z = 1$ and radius 1.

Next question on next page

SECTION B *Each question carries 20 marks. You may attempt all questions but only marks for the best TWO questions will be counted.*

B6.

(i) Prove that if the complex function $f = u + iv$ is differentiable at $z = x + iy$ then the real and imaginary parts u and v of f satisfy the *Cauchy-Riemann equations* there, and find an expression for the derivative of f at z in terms of the values of the partial derivatives of u and v there.

(ii) Define what it means to sav that a real-valued function on the plane $\mathbf{R}^2$ is *harmonic*, and prove that the real and imaginary parts u and v of a homomorphic function f are harmonic. *(You may assume that u and v have partial derivatives of all orders at z.)*

(iii) Let

$$u(x, y) = e^y \cos x.$$

Show that u is harmonic on $\mathbf{R}^2$ and find a homomorphic function f such that u is the real part of f.

B7.

(i) State *Cauchy's Integral Formula* and use it to prove that if f is homomorphic on and everywhere inside a circle having centre z_0 and radius r then

$$f(z_0) = \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + re^{i\theta})d\theta$$

(Gauss' Mean Value Theorem).

(ii) What is meant by an *isolated singularity* of a complex function f ? Explain how such singularities may be classified into types. Find all singularities, and their types, for the function

$$f(z) = z \cot z.$$

B8.

State and prove the *Fundamental Theorem of Algebra.* (You should state in full any theorems you need to assume in the proof.)

How many zeros (counted with multiplicity) does the polynomial

$$p(z) = 2z^5 - 6z^4 + 12z - 3$$

have

(i) in the complex plane C ?

(ii) inside the circle $|z| = 2$?

(iii) in the annulus $\{z : 1 < |z| < 2\}$?

Give reasons for your answers.

B9.

(i) Evaluate

$$\int_C \frac{1}{(z^2 - 4)^2} dz$$

where C is the (positively oriented) circle having centre $z = 2$ and radius 2.

(ii) Evaluate the real integral

$$\int_{-\infty}^{\infty} \frac{(x+-1)^2}{(x^2+1)(x^2+4)} dx.$$

End of examination paper