Queen Mary and Westfield College
University of London
BSc Examination 2000

MAS205 Complex Variables

`??? May 2000 10.00am`

The duration of this examination is 2 hours.

This paper has two Sections and you should attempt both Sections. Please read carefully the instructions given at the beginning of each Section.

Calculators may be used in this examination, but any programming, graph plotting or algebraic facility may not be used. Please state on your answer book the name and type of machine used.

SECTION A *Each question carries 12 marks. You should attempt ALL questions.*

A1.
(i) Find all solutions $z \in \mathbf{C}$ of the equation

$$z^5 + 32 = 0.$$

(ii) Find all solutions $z \in \mathbf{C}$ of the equation

$$e^{2z} = 1$$

and hence or otherwise find all solutions $z \in \mathbf{C}$ of the equation

$$\sinh(z) = 0.$$

A2.
(i) Evaluate

$$(a) \quad \lim_{z \to 2i} \frac{z^2 - 5iz - 6}{z^2 + 4} \qquad\qquad (b) \quad \lim_{z \to \infty} \frac{(1-2z)(1+2z)}{1+iz^2}$$

(ii) Write down the *Cauchy-Riemann equations* satisfied by the real and imaginary parts u and v of a complex function $f = u + iv$ at any point z_0 where f is complex differentiable. If u and v satisfy the Cauchy-Riemann equations at z_0 what extra condition on u and v will ensure that f is complex differentiable at z_0 ?

Let $f(z) = y(3x^2 - y^2) + ix(x^2 - 3y^2)$. Show that f is complex differentiable at just one point, and compute its derivative at this point.

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A3.
(i) Let

$$f(z) = \frac{z+1}{z+2}.$$

Find the Taylor series $\sum_0^\infty a_n(z+1)^n$ for f about the point $z = -1$.
For what values of z does this series converge absolutely?

(ii) Let

$$g(z) = \frac{1}{(z-1)(3-z)}.$$

Find the Laurent series $\sum_0^\infty a_n(z-1)^n + \sum_1^\infty b_n(z-1)^{-n}$ for g about the point $z = 1$.
For what values of z does this series converge absolutely?

A4. Let C be a *contour* parametrised by a piecewise smooth function $\gamma : [a, b] \to \mathbf{C}$. Define what is meant by the *contour integral*

$$\int_C f(z)dz$$

of the complex function f along the contour C. Evaluate this integral when $f(z) = \bar{z}$ (the complex conjugate of z) and

(i) C is the straight line path from $z = +1$ to $z = -1$;

(ii) C is the upper half of the unit circle, from $z = +1$ to $z = -1$.

Is it possible that the function $f(z) = \bar{z}$ has an *antiderivative* on $\mathbf{C}$? Find one or else give a reason why such an antiderivative cannot exist.

A5.
Evaluate

$$\int_C \frac{e^{\pi i z}}{(z-1)(z+2)} dz$$

in each of the following two cases:

(i) C is the positively oriented circle with centre $z = 1$ and radius 2;

(ii) C is the positively oriented circle with centre $z = -1/2$ and radius 1.

SECTION B *Each question carries 20 marks. You may attempt all questions but only marks for the best TWO questions will be counted.*

B6.
(i) Consider the transformation $z = x + iy \to w = u + iv$ defined by

$$w = z^2 + 1.$$

Show that the image of the line $x = \alpha$ (where α is a real constant) has equation

$$u = \alpha^2 + 1 - (v/2\alpha)^2.$$

Find the equation of the image of the line $y = \beta$ (where β is a real constant). Sketch the image of the square which has vertices $z = 0, z = 1, z = 1 + i$ and $z = i$.

(ii) What does it mean to say that a map $M : z \to w$ is a *fractional linear* (or *Möbius*) transformation ? Find the Möbius transformation M which sends $z = -1$ to $w = 2$, $z = 0$ to $w = 2i$ and $z = 1$ to $w = -2$. Under this transformation M, what is the image in the w-plane of the real axis in the z-plane ? What is the image of the upper half of the complex z-plane ?

B7.
What does it mean to say that a complex-valued function on a domain $U \subset \mathbf{C}$ is *holomorphic* ?

(i) What does it mean to say that a real-valued function on a domain $U \subset \mathbf{R}^2$ is *harmonic*? Prove that if f is a complex-valued function which is holomorphic on a domain $U \subset \mathbf{C}$ then its real and imaginary parts u and v are harmonic functions of (x, y) (where $z = x + iy$). *(You may assume that u and v satisfy the Cauchy-Riemann equations at z and that u and v have continuous second order partial derivatives at z.)* Let

$$u(x, y) = y^2 - x^2 + y - x.$$

Show that u is harmonic on $\mathbf{R}^2$ and find a holomorphic function f such that u is the real part of f.

(ii) State and prove *Cauchy's Integral Formula* for the value $f(z_0)$ of a holomorphic function f at a point z_0 inside a simple closed contour C.

B8.
What is meant by an *isolated singularity* of a complex function f ?
What does it mean to say that such a singularity is a *pole of order* m ?
What is meant by the *residue* of f at an isolated singularity ?

Prove that if a holomorphic function f has a zero of multiplicity m at z_0 then the function $\Phi(z) = f'(z)/f(z)$ has a simple pole at z_0, with residue m there. State the *Residue Theorem* and deduce the value of

$$\int_C f'(z)/f(z) dz$$

where C is a positively oriented simple closed contour and f is a holomorphic function having finitely many zeros inside C (and none on C itself).

State Rouché's Theorem (without proof). How many zeros (counted with multiplicity) does the polynomial

$$p(z) = z^5 - 4z^3 + 2$$

have in the annulus $\{z : 1 < |z| < 3\}$?

B9.
(i) Evaluate

$$\int_C \frac{1}{z^2(z^2-1)} dz$$

where C is the positively oriented circle having centre $z = 1/2$ and radius 1.

(ii) Evaluate the real integral

$$\int_{-\infty}^{\infty} \frac{1}{(x^2+1)(x^2+x+1)} dx.$$

End of examination paper