Queen Mary
University of London
BSc Examination 2002

MAS205 Complex Variables

13th May 2002 2.30pm

The duration of this examination is 2 hours.

This paper has two Sections and you should attempt both Sections. Please read carefully the instructions given at the beginning of each Section.

Calculators are NOT permitted in this examination.

SECTION A *Each question carries 12 marks. You should attempt ALL questions.*

A1.
(i) Write down the *Cauchy-Riemann equations* satisfied by the real and imaginary parts u and v of a complex function $f = u + iv$ at any point z_0 where f is complex differentiable. If u and v satisfy the Cauchy-Riemann equations at z_0 what extra condition on u and v will ensure that f is complex differentiable at z_0 ?

(ii) Determine at which points each of the following functions f is differentiable.
(*a*) $f(x+iy) = x^2 - y^2 - y + ix(2y+1)$.
(*b*) $f(x+iy) = 6xy + i(3x + 2y^3)$,
(*c*) $f(x+iy) = y^3 - 3x^2y + i(3xy^2 - x^3)$,

A2.
(a) Consider the transformation $z \mapsto w = 1 - z^2$. If $w = u + iv$, show that the image of the line $x = c$ (for c a real constant) has equation

$$u = 1 - c^2 + \left(\frac{v}{2c}\right)^2 .$$

(b) Find the equation of the image of the line $y = c$, if c is a real constant.

(c) Sketch the images in the w-plane of the lines $x = 1$ and $y = 3$, and determine those complex numbers w at which the two images intersect.

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A3.

(a) Find the five solutions $z_1, z_2, z_3, z_4, z_5 \in \mathbb{C}$ of the equation $z^5 + 32 = 0$.

(b) Evaluate $|z_1 + z_2 + z_3 + z_4 + z_5|$.

(c) Use your answer to (a) above, together with Rouché's Theorem, to determine the number of zeros (counted with algebraic multiplicity) of the polynomial $p(z) = z^5 + 3z^2 + 32$ which lie inside the disc $\{z \in \mathbb{C} : |z| < 3\}$.

A4.

(a) Determine the Laurent series $\sum_{n=0}^{\infty} a_n (z-1)^n + \sum_{n=1}^{\infty} b_n (z-1)^{-n}$ of the function $f(z) = \frac{1}{(z-1)(z+3)}$ on a punctured disc centred at $z_0 = 1$.

(b) For what values of z is this Laurent series valid?

(c) What type of singularity does f have at the point $z_0 = 1$?

(d) Determine the residue of f at the point $z_0 = 1$.

A5.

(i) Let C be a *contour* parametrised by a piecewise smooth function $\gamma : [a, b] \to \mathbb{C}$. Define what is meant by the *contour integral*

$$\int_C f(z)\, dz$$

of the complex function f along the contour C.
Evaluate this integral when $f(z) = \bar{z}$ (the complex conjugate of z) and C is the straight line segment from i to $-i$.

(ii) State Cauchy's Theorem.
Let C be a positively oriented regular hexagon with vertices at $z_j = 4 + 3e^{2\pi i j/6}$, for $j = 0, \ldots, 5$. Use Cauchy's Theorem, verifying carefully that all its hypotheses are satisfied, to evaluate the integral

$$\int_C \frac{e^z}{z - 100}\, dz\ .$$

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SECTION B *Each question carries 20 marks. You may attempt all questions but only marks for the best TWO questions will be counted.*

B6.

(i) What does it mean to say that a function $f : \mathbb{C} \to \mathbb{C}$ is *differentiable* at a point $z_0 \in \mathbb{C}$?
What does it mean to say that a function $f : \mathbb{C} \to \mathbb{C}$ is *holomorphic* on a domain U?
What does it mean to say that a function $f : \mathbb{C} \to \mathbb{C}$ is *entire*?
Give an example of a function which is holomorphic in the unit disc $\{z \in \mathbb{C} : |z| < 1\}$ but is not entire.

(ii) Suppose $f : \mathbb{C} \to \mathbb{C}$ is entire. Suppose $z_0 \in \mathbb{C}$, $R > 0$ is real, and $n \geq 0$ is an integer. Use the extended version of Cauchy's Integral Formula ($f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_C \frac{f(z)}{(z-z_0)^{n+1}}\, dz$, where C is any positively oriented circle centred at z_0) to prove that

$$|f^{(n)}(z_0)| \leq \frac{n!}{R^n} \max\{|f(z)| : |z - z_0| = R\} \; .$$

(iii) Use the case $n = 1$ of the inequality in (ii) to prove Liouville's Theorem: If $f : \mathbb{C} \to \mathbb{C}$ is entire, and there exists $K > 0$ such that $|f(z)| \leq K$ for all $z \in \mathbb{C}$, then f is a constant function.

B7.

(i) Write down the definition of the *radius of convergence* of a power series $\sum_{n=0}^{\infty} a_n (z - z_0)^n$.
Prove that such a radius of convergence always exists.

(ii) Find the Taylor series expansion $\sum_{n=0}^{\infty} a_n (z - 3)^n$ of the function $f(z) = 1/(1 + 4z)$ about the point $z_0 = 3$.

What is the radius of convergence of this Taylor series?

(iii) Suppose the power series $\sum_{n=0}^{\infty} b_n z^n$ has radius of convergence $R < \infty$, and the power series $\sum_{n=0}^{\infty} c_n z^n$ has radius of convergence $R' < \infty$.
State whether it is True or False that the power series $\sum_{n=0}^{\infty} (b_n + c_n) z^n$ has radius of convergence equal to $\min(R, R')$.
Either prove this result, or disprove it by giving a counterexample.

B8.

(i) What is meant by an *isolated singularity* of a complex function f ?
What does it mean to say that such a singularity is a *pole of order* m ?
What is meant by the *residue* of f at an isolated singularity ?

Prove that if a holomorphic function f has a zero of multiplicity m at z_0 then the function $\Phi(z) = f'(z)/f(z)$ has a simple pole at z_0, with residue m there. State the *Residue Theorem* and deduce the value of

$$\int_C f'(z)/f(z)\,dz$$

where C is a positively oriented simple closed contour and f is a holomorphic function having finitely many zeros inside C (and none on C itself).

(ii) State Rouché's Theorem (without proof). How many zeros (counted with multiplicity) does the polynomial $p(z) = z^6 - 5z^5 + 10z^4 - 2z^2 + 1$ have in the annulus $\{z \in \mathbb{C} : |z| \geq 1\}$?

B9.

(i) Use the Residue Theorem to evaluate

$$\int_C \frac{1}{z^2(z^2-9)}dz$$

where C is the positively oriented circle having centre $z = -1$ and radius 3.

(ii) Use the Residue Theorem to prove that

$$\int_{-\infty}^{\infty} \frac{1}{(x^2+4)^2}\,dx = \frac{\pi}{16}\ .$$

Deduce the value of

$$\int_0^{\infty} \frac{1}{(x^2+4)^2}\,dx.$$

End of examination paper