Queen Mary
University of London
BSc Examination 2003 BY COURSE UNIT

MAS205 Complex Variables

`23rd May 2003    2.30pm`

The duration of this examination is 2 hours.

This paper has two Sections and you should attempt both Sections. Please read carefully the instructions given at the beginning of each Section.

Calculators are NOT permitted in this examination. The unauthorised use of a calculator constitutes an examination offence.

SECTION A *Each question carries 12 marks. You should attempt ALL questions.*

A1.
(i) Find all solutions $z \in \mathbb{C}$ of the equation

$$z^4 + 16 = 0.$$

(ii) Find all solutions $z \in \mathbb{C}$ of the equation

$$e^{2z} = -1.$$

Hence determine all solutions of the equation $\cosh(z) = 0$.
What are the solutions of $\sinh(z) = 0$?

A2.
(i) Consider the transformation $z \mapsto w = z^4$. Determine, and sketch, the region in the z-plane which is mapped to the upper-half $\{w : \operatorname{Im}(w) \geq 0\}$ of the w-plane.

(ii) Write down the *Cauchy-Riemann equations* satisfied by the real and imaginary parts u and v of a complex function $f = u + iv$ at any point z_0 where f is complex differentiable. If u and v satisfy the Cauchy-Riemann equations at z_0 what extra condition on u and v will ensure that f is complex differentiable at z_0 ?
Let $f(z) = y(3x^2 - y^2) + ix(x^2 - 3y^2)$. Show that f is complex differentiable at just one point, and compute its derivative at this point.

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A3.

(i) Let

$$f(z) = \frac{az+b}{z^2+4} \qquad \text{for some } a, b \in \mathbb{C}.$$

If $\lim_{z\to 2i} f(z) = 3$ then determine the coefficients a and b.

(ii) Let

$$f(z) = \frac{z+3}{z+1}.$$

Find the Taylor series $\sum_{n=0}^{\infty} a_n(z+3)^n$ for f about the point $z=-3$.
Determine the value of the coefficient a_3 in this Taylor series.
For what values of z does this Taylor series converge absolutely?

A4. Let C be a *contour* parametrised by a piecewise smooth function $\gamma : [a,b] \to \mathbb{C}$. Define what is meant by the *contour integral*

$$\int_C f(z)dz$$

of the complex function f along the contour C. Evaluate this integral when $f(z) = \bar{z}$ (the complex conjugate of z) and

(i) C is the straight line path from $z=+1$ to $z=-1$;

(ii) C is the upper half of the unit circle, from $z=+1$ to $z=-1$.

Is it possible that the function $f(z) = \bar{z}$ has an *antiderivative* on $\mathbb{C}$? Find one or else give a reason why such an antiderivative cannot exist.

A5.

Evaluate

$$\int_C \frac{e^{\pi i z}}{(z-1)(z+2)} dz$$

in each of the following two cases:

(i) C is the positively oriented circle with centre $z=3$ and radius 4;

(ii) C is the positively oriented circle with centre $z=0$ and radius 3.

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SECTION B *Each question carries 20 marks. You may attempt all questions. Except for the award of a bare pass, only marks for the best TWO questions will be counted.*

B6.
(i) Consider the transformation $z = x + iy \mapsto w = u + iv$ defined by

$$w = z^2 + 1.$$

Show that the image of the line $x = c$ (where c is a non-zero real constant) has equation

$$u = c^2 + 1 - \left(\frac{v}{2c}\right)^2.$$

Find the equation of the image of the line $y = d$ (where d is a non-zero real constant).
Sketch the images in the w-plane of the lines $x = 1$ and $y = 1$.
At what points in the w-plane do the images of these lines intersect?

(ii) What does it mean to say that a map $f : z \mapsto w$ is a *Möbius* (or *fractional linear*) transformation ? Find the Möbius transformation f which sends $z = -1$ to $w = 2$, $z = 0$ to $w = 2i$ and $z = 1$ to $w = -2$.
Under this transformation f, what is the image in the w-plane of the real axis in the z-plane ?
What is the image of the upper half of the complex z-plane ?

B7.
(i) What does it mean to say that a function $f : \mathbb{C} \to \mathbb{C}$ is *entire*?
State Liouville's Theorem, and use it to show that if

$$P(z) = a_0 + a_1 z + \ldots + a_n z^n$$

for some $n \geq 1$, where $a_j \in \mathbb{C}$ for all $0 \leq j \leq n$, and $a_n \neq 0$, then P has at least one zero in $\mathbb{C}$.

(ii) State Rouché's Theorem. Let

$$P(z) = z^4 - z^3 + 9z^2 - 2z + 7.$$

Use Rouché's Theorem to determine the number of zeros of P (counted with multiplicity) which lie inside the circle of radius 2 centred at 0.

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B8.

(i) State Cauchy's Theorem. By applying Cauchy's Theorem, or otherwise, show that

$$\int_C \frac{1}{z^2 - 6z + 10}\, dz = 0\,,$$

where $C = \{z \in \mathbb{C} : |z| = 1\}$ is the unit circle.

(ii) State the Deformation Principle, and briefly explain how Cauchy's Theorem can be used to prove it.

State Cauchy's Integral Formula for the value $\varphi(z_0)$ of a holomorphic function φ at a point z_0 inside a simple closed contour C.

Use the Deformation Principle to prove Cauchy's Integral Formula.

B9.

(i) What does it mean to say that a point z_0 is an *isolated singularity* of a complex function f ?

What does it mean to say that such a singularity is a *pole of order* m ?

What is meant by the *residue* of f at an isolated singularity ?

(ii) Find the Laurent series $\sum_{n=0}^{\infty} a_n(z+1)^n + \sum_{n=1}^{\infty} b_n(z+1)^{-n}$ of the function

$$f(z) = \frac{1}{(z-2)(z+1)^3}$$

on a punctured disc centred at the point $z_0 = -1$.

For what values of z is this Laurent series absolutely convergent?

What is the order of the pole at $z_0 = -1$?

What is the residue of f at $z_0 = -1$?

(iii) Suppose the complex functions g, h both have simple poles at some point $z_0 \in \mathbb{C}$.

Does gh have a pole at z_0? If so, what is its order?

Does $g + h$ have a pole at z_0? If so, what is its order?

End of examination paper