MAS205 Complex Variables 2004-2005

Exercises 1

Exercise 1: Let $z_1 = 2 + i$ and $z_2 = 3 - 2i$. Compute (in standard $x + iy$ form):

$$(a)\quad z_1 z_2 \qquad (b)\quad \frac{1}{z_1} \qquad (c)\quad \frac{z_2}{z_1} \qquad (d)\quad \frac{1}{z_1} + \frac{1}{z_2}$$

Compute the moduli:

$$(a)\quad |z_1| \qquad (b)\quad \left|\frac{z_1}{z_2}\right| \qquad (c)\quad |z_1 z_2|$$

Exercise 2: Express the following complex numbers in polar exponential form:

$$(a)\quad 1 \qquad (b)\quad -2i \qquad (c)\quad 1 - i \qquad (d)\quad \sqrt{3} - i \qquad (e)\quad (1+i)^2$$

Exercise 3: Solve for the roots of the following equations:

(a) $z^3 + 8 = 0$

(c) $(z+1)^4 - 1 = 0$

Express all the roots in standard and polar form, and draw diagrams showing their location in the complex plane.

Exercise 4: Describe graphically the sets of points in the complex plane defined by the following equations and inequalities:

(a) $|z - 3 - 2i| < 3$

(b) $\Im(z^3) = 0$

(c) $1 \leq \Re(z + i) < 2$

(d) $z^2 = -4$

Notation: $\Re(z)$ and $\Im(z)$ denote the real and imaginary parts of z, respectively.

Please hand in your solutions (to the yellow Complex Variables box on the ground floor) by 11am Tuesday 12th October

Thomas Prellberg, September 2004

25 marks each

1) $z_1 = 2+i \quad , \quad z_2 = 3-2i$

(a) $z_1 z_2 = (2+i)(3-2i) = 6+2+3i-4i = 8-i$ 4

(b) $\dfrac{1}{z_1} = \dfrac{1}{2+i} = \dfrac{2-i}{4+1} = \dfrac{2}{5} - \dfrac{1}{5}i$ 4

c) $\dfrac{z_2}{z_1} = \dfrac{1}{z_1} z_2 = \dfrac{(2-i)(3-2i)}{5} = \dfrac{6-2-3i-4i}{5}$ 4

$= \dfrac{4}{5} - \dfrac{7}{5}i$

d) $\dfrac{1}{z_2} = \dfrac{1}{3-2i} = \dfrac{3+2i}{9+4} = \dfrac{3}{13} + \dfrac{2}{13}i$ 4

$\rightsquigarrow \dfrac{1}{z_1} + \dfrac{1}{z_2} = \left(\dfrac{2}{5} + \dfrac{3}{13}\right) + \left(-\dfrac{1}{5} + \dfrac{2}{13}\right)i = \dfrac{41}{65} - \dfrac{3}{65}i$

(a) $|z_1| = |2+i| = \sqrt{4+1} = \sqrt{5}$ 3

(b) $\left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|} = \dfrac{|2+i|}{|3-2i|} = \sqrt{\dfrac{4+1}{9+4}} = \sqrt{\dfrac{5}{13}}$ 3

(c) $|z_1 z_2| = |z_1||z_2| = \sqrt{5 \cdot 13} = \sqrt{65}$ $\quad (= |8-i|)$ 3

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2) (a) $1 = 1e^{0i}$ 5

(b) $-2i = 2\, e^{-i\frac{\pi}{2}}$ 5

(c) $1-i = \sqrt{2}\, e^{-i\frac{\pi}{4}}$ 5

(d) $\sqrt{3} - i = 2\, e^{-i\frac{\pi}{6}}$ 5

(e) $(1+i)^2 = 2\, e^{i\frac{\pi}{2}}$ $\quad$ (either $\left(\sqrt{2}\, e^{i\frac{\pi}{4}}\right)^2$ or $(1+i)^2 = 2i$) 5

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3) (a) $z^3 = -8 \rightsquigarrow z = \left(8 e^{i\pi + 2k\pi i}\right)^{1/3}$ 4

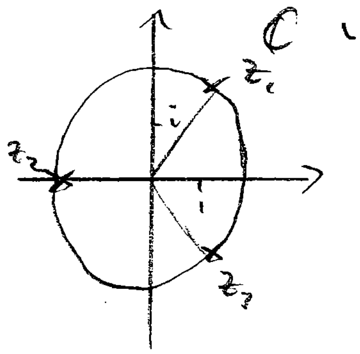
$\rightsquigarrow z = 2 e^{(2k+1) i \frac{\pi}{3}}$ $k = 0, 1, 2$

$z_1 = 2 e^{i\frac{\pi}{3}} = 1 + \sqrt{3} i$

$z_2 = 2 e^{i\pi} = -2$

$z_3 = 2 e^{\frac{5 i \pi}{3}} = 1 - \sqrt{3} i$

3 4



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(b) $(z+1)^4 - 1 = 0$

$\rightsquigarrow z = e^{i\frac{k\pi}{4}} - 1$ $k = 0, 1, 2, 3$

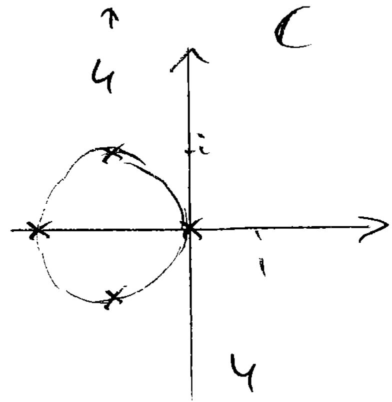
$z_1 = e^{i0} - 1 = 0$ indeterminate polar form

$z_2 = e^{i\frac{\pi}{4}} - 1 = i - 1 = \sqrt{2} e^{i\frac{3\pi}{4}}$

$z_3 = e^{\frac{i\pi}{2}} - 1 = -1 - 1 = -2 = 2 e^{i\pi}$

$z_4 = e^{i\frac{3\pi}{4}} - 1 = -i - 1 = \sqrt{2} e^{-i\frac{3\pi}{4}}$

3 3 4



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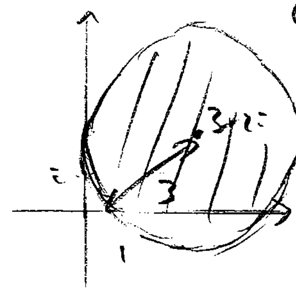
4) (a) $|z-3-2i| < 3$

open disk centered at $3+2i$ with radius 3



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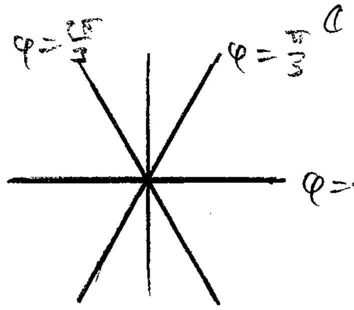
(b) $\mathrm{Im}(z^3) = 0$



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$z = re^{i\varphi}$

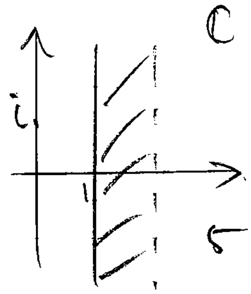
$z^3 = r^3 e^{3i\varphi}$

$\mathrm{Im}(z^3) = 0 \rightsquigarrow 3\varphi = k\pi \quad k \in \mathbb{Z}$

$\rightsquigarrow \varphi = 0, \frac{\pi}{3}, \frac{2\pi}{3}, \pi, \frac{4\pi}{3}, \frac{5\pi}{3}$

lines through the origin angled at $\varphi = 0, \pm 60°$

(c) $1 \leq \mathrm{Re}(z+i) < 2 \rightsquigarrow 1 \leq \mathrm{Re}(z) < 2$

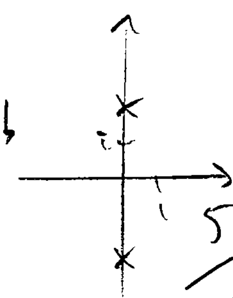


$\rightsquigarrow \mathrm{Re}(z) \in [1, 2)$

5

Vertical strip between 1 and 2, closed to the left, open to the right

(d) $z^2 = -4 \rightsquigarrow z = \pm 2i$ two points



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