MAS205 Complex Variables 2004-2005

Exercises 2

Exercise 5: Find all complex solutions of the following equations:

$$(a)\quad e^z = 1 \qquad (b)\quad e^{2z} = -1 \qquad (c)\quad \cosh z = 0 \qquad (d)\quad \sin z = 0$$

Exercise 6: Consider the transformation

$$z \mapsto w = iz^2 + 1\ .$$

(a) Find the equation of the image of the line $\Im(z) = 1$ and sketch the image.

(b) Sketch the image of the curve $z\bar{z} = 1$.

Exercise 7: For each of the following transformations, find the regions in the z-plane which map to the left half of the w-plane:

(a) $w = 1 + 1/z$

(b) $w = z^3$

Exercise 8: Find the Möbius transformation $f(z) = (az + b)/(cz + d)$ which maps $1 \mapsto 1$, $i \mapsto 0$, and $-1 \mapsto i$.

(a) What is the image of $z = 0$

(b) Which point is mapped by f to $-i$?

(c) What is the image of the left half plane under f?

Exercise 9: Prove that if $g(z) = (az + b)/(cz + d)$ and $h(z) = (a'z + b')/(c'z + d')$, then $h \circ g(z) = (a''z + b'')/(c''z + d'')$ where

$$\begin{pmatrix} a'' & b'' \\ c'' & d'' \end{pmatrix} = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Please hand in your solutions (to the yellow Complex Variables box on the ground floor) by 11am Tuesday 19th October

Thomas Prellberg, October 2004

20 marks each

5) (a) $e^z = 1$, $z = x+iy$ $\Big\}$ $e^x e^{iy} = 1$ 5

polar form: $e^x = 1$, $y = 2k\pi$, $k \in \mathbb{Z}$

$\rightarrow$ $\underline{z = 2k\pi i, \; k \in \mathbb{Z}}$

(b) $e^{2z} = -1$, $z = x+iy$ $\Big\}$ $e^{2x} e^{2iy} = -1 = e^{i\pi}$ 5

$\rightsquigarrow$ $e^x = 1$, $2y = \pi + 2k\pi$, $k \in \mathbb{Z}$

$\rightsquigarrow$ $\underline{z = \frac{2k+1}{2} \pi i \quad k \in \mathbb{Z}}$

(c) $\cosh z = 0 \iff e^z + e^{-z} = 0$

$(e^z \neq 0 \; \forall z \in \mathbb{C}) \iff e^{2z} + 1 = 0$ 5

reduced to (b) $\underline{z = \frac{2k+1}{2} \pi i, \; k \in \mathbb{Z}}$

(d) $\sin z = 0 \iff e^{iz} - e^{-iz} = 0$

$\iff e^{2iz} = 1$ 5

by (a) $2iz = 2k\pi i$, $k \in \mathbb{Z}$ $\rightsquigarrow$ $\underline{z = k\pi, \; k \in \mathbb{Z}}$

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6) $w = iz^2 + 1 = i(x^2 - y^2 + 2ixy) + 1$

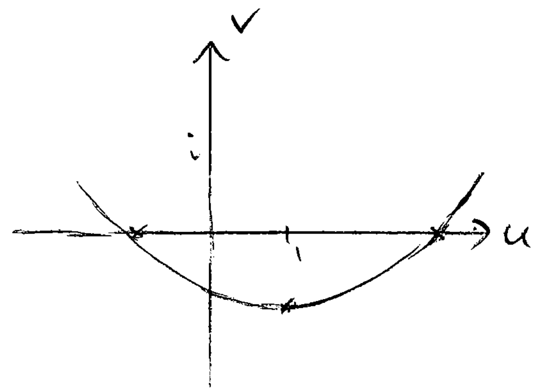
$u = 1 - 2xy \quad , \quad v = x^2 - y^2$

(a) $\mathrm{Im}(z) = 1 \quad \rightsquigarrow \quad z = x + i \, , \quad x \in \mathbb{R}$

i.e. $y = 1$

$\rightsquigarrow \quad u = 1 - 2x \, , \quad v = x^2 - 1$

$x = \frac{1-u}{2} \quad , \quad v = \left(\frac{1-u}{2}\right)^2 - 1$ parabola 5



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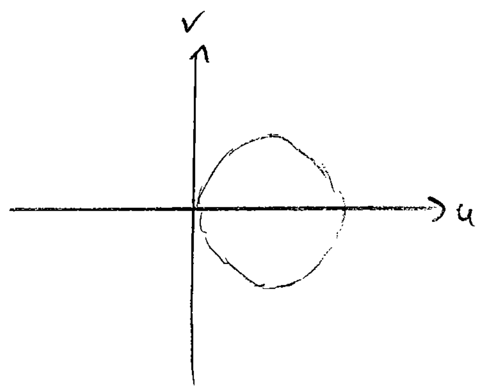
(b) $z\bar{z} = 1 \iff |z| = 1 \, , \quad z = e^{i\varphi}$

$w = iz^2 + 1 = 1 + e^{i(\pi + 2\varphi)}$ 5

circle with radius one centred at 1



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$\begin{pmatrix} \pm 1 \mapsto 1+i \\ \pm i \mapsto 1-i \end{pmatrix}$

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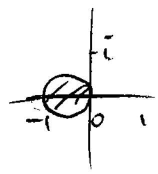
7)

(a) $w = 1 + \frac{1}{z} = 1 + \frac{x}{x^2+y^2} + i\,\frac{-y}{x^2+y^2}$ 5

$\mathrm{Re}(w) \leq 0 \iff 1 + \frac{x}{x^2+y^2} \leq 0$

$\iff x^2 + y^2 + x \leq 0 \iff \underline{\left(x+\frac{1}{2}\right)^2 + y^2 \leq \left(\frac{1}{2}\right)^2}$

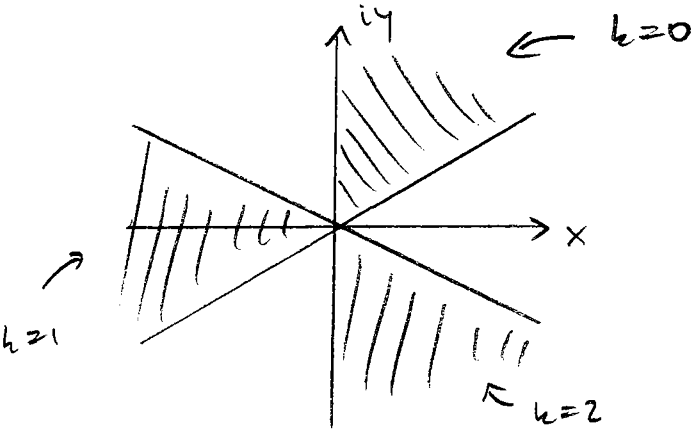
disk centred at $z = -\frac{1}{2}$ with radius $\frac{1}{2}$ 5



(b) $w = z^3 = r^3 e^{3i\varphi}$

$\mathrm{Re}(w) \leq 0 \iff \frac{\pi}{2} + 2k\pi \leq 3\varphi \leq \frac{3\pi}{2} + 2k\pi$

$\frac{\pi}{6} + \frac{2}{3}k\pi \leq \varphi \leq \frac{\pi}{2} + \frac{2}{3}k\pi$ 5



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8) (i) $\dfrac{a+b}{c+d}=1 \quad \rightsquigarrow \quad a+b=c+d$

(ii) $\dfrac{ai+b}{ci+d}=0 \quad \rightsquigarrow \quad b=ia$

(iii) $\dfrac{-a+b}{-c+d}=i \quad \rightsquigarrow \quad -a+b=-ic+id$

$$-ia+\underbrace{ib}_{a} = c-d$$

$$\left.\begin{aligned}(1-i)\,a &= c+d\\ (1-i)\,a &= c-d\end{aligned}\right\}\quad c=(1-i)\,a\,,\quad d=0$$

$$\rightsquigarrow \quad f(z)=\frac{z-i}{(1-i)z}$$ 8

check: $f(1)=\dfrac{1-i}{(1-i)1}=1$ ✓ $f(i)=\dfrac{i-i}{(1-i)i}=0$ ✓

$f(-1)=\dfrac{-1-i}{(1-i)(-1)}=\dfrac{1+i}{1-i}=\dfrac{(1+i)^2}{2}=i$ ✓

(a) $f(0)=\infty$ 4

(b) $\dfrac{z-i}{(1-i)z}=-i \quad \rightarrow \quad z-i=(-i)(1-i)\,z=(-i-1)\,z$

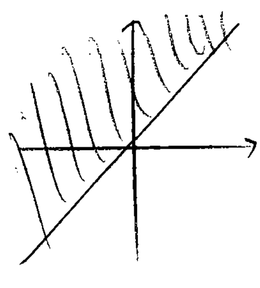
$$(2+i)\,z=i$$ 4

$z=\dfrac{1+2i}{5}$

(c) $f(0)=\infty,\quad f(i)=0,\quad f(-i)=\dfrac{-i-i}{(1-i)(-i)}=\dfrac{2}{1-i}=1+i$

Image of imaginary axis is line through zero under 45° 4

$f(1)=1$



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9)

$$g(z) = \frac{az+b}{cz+d}$$

$$h(z) = \frac{a'z+b'}{c'z+d'}$$

$$h\circ g(z) = \frac{a'\frac{az+b}{cz+d}+b'}{c'\frac{az+b}{cz+d}+d'} = \frac{(a'a+b'c)z+(a'b+b'd)}{(c'a+d'c)z+(c'b+d'd)}$$ 10

compare with

$$\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a'a+b'c & a'b+b'd \\ c'a+d'c & c'b+d'd \end{pmatrix}$$ 10

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