MAS205 Complex Variables 2004-2005

Exercises 3

Exercise 10: Evaluate the following limits:

$$\text{(a)} \quad \lim_{z\to\infty} \frac{((2+i)z+1)(z+3)^3}{(2z-i)^2(3z-4)^2} \qquad \text{(b)} \quad \lim_{z\to 1+i} \frac{z^6}{z^2-2i} \qquad \text{(c)} \quad \lim_{z\to\infty} \frac{z^2}{z^3-1-i}$$

Exercise 11: (a) Give an example of a function $f : \mathbb{C} \to \mathbb{C}$ such that

$$\lim_{z\to i} f(z) = 2 \qquad \text{and} \qquad \lim_{z\to 1} f(z) = \infty\,.$$

(b) Suppose

$$f(z) = \frac{p(z)}{z^2-1}\,, \quad \text{where } p(z) = az+b \text{ for some } a, b \in \mathbb{C}.$$

If $\lim_{z\to 1} f(z) = 1$, what is $p(z)$?

(c) Suppose

$$f(z) = \frac{p(z)}{z^2+1}\,, \quad \text{where } p(z) \text{ is a quadratic polynomial.}$$

If $\lim_{z\to i} f(z) = i$ and $\lim_{z\to\infty} f(z) = 2$, what is $p(z)$?

(d) Find a polynomial $p(z)$ such that

$$\lim_{z\to 0} \frac{p(z)}{z(z-i)} = 3i\,, \quad \lim_{z\to -i} \frac{p(z)}{z(z-i)} = 0\,, \quad \lim_{z\to 3+i} \frac{p(z)}{z(z-i)} = 0\,.$$

Exercise 12: For each of the following functions, decide at which values of z the function is continuous and at which values it is not continuous. Give reasons, but detailed proofs are not expected.

(a) $f(z) = z^2 + i\overline{z} - 1$

(b) $f(z) = i(\overline{z}/z)^4$ for all non-zero z, and $f(0) = i$.

Exercise 13: Starting from the definition of the derivative of a complex function as a limit,

(a) find the derivative of $f(z) = z^3 - 2z$ at $z = i$;

(b) find the derivative of $f(z) = z^2 - 1$ for all $z \in \mathbb{C}$;

(c) prove that $f(z) = z\overline{z} - 2z$ does not have a derivative at z_0 unless $z_0 = 0$. What is the value of $f'(0)$?

Please hand in your solutions (to the yellow Complex Variables box on the ground floor) by 11am Tuesday 26th October

Thomas Prellberg, October 2004

25 each

10)

(a)
$$\lim_{z\to\infty} \frac{((2+i)z+1)(z+3)^3}{(2z-i)^2(3z-4)^2}$$

$$= \lim_{z\to\infty} \frac{\left(2+i+\frac{1}{z}\right)\left(1+\frac{3}{z}\right)^3}{\left(2-\frac{i}{z}\right)^2\left(3-\frac{4}{z}\right)^2}$$

$$= \frac{(2+i)(1)^3}{2^2\ 3^2} = \frac{2+i}{36}$$

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(b)
$$\lim_{z\to 1+i} \frac{z^6}{z^2-2i} = \infty ,$$

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as $\lim_{z\to 1+i} z^6 = (1+i)^6 = (2i)^3 = -8i$

and $\lim_{z\to 1+i} z^2-2i = 2i-2i = 0$

(c)
$$\lim_{z\to\infty} \frac{z^2}{z^3-1-i} = \lim_{z\to\infty} \frac{1/z}{1-\frac{1+i}{z^3}} = \frac{0}{1} = 0.$$

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11)

(a) e.g. $f(z) = \dfrac{z-i}{z-1} + 2$

check: rational function, denom. zero at $z=1$ ✓

$f(i) = \dfrac{i-i}{i-1} + 2 = 2$ ✓

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(b) $f(z) = \dfrac{az+b}{z^2-1} = \dfrac{p(z)}{q(z)}$

$\lim_{z\to 1} f(z) = 1$ possible only if $p(1) = 0$, as $q(1) = 0$ 6

$\rightsquigarrow$ $b = -a$ and $f(z) = \dfrac{a\cancel{(z-1)}}{\cancel{(z-1)}(z+1)} = \dfrac{a}{z+1}$

$1 = \lim_{z\to 1} \dfrac{a}{z+1} = \dfrac{a}{2} \rightsquigarrow a = 2$ and thus $\underline{p(z) = 2z-2}$

(c) $f(z) = \dfrac{p(z)}{z^2+1}$ with $p(z)$ quadratic

$\lim_{z\to i} f(z) = i$ possible only if $p(i) = 0$, as $q(i) = 0$

$\rightsquigarrow p(z) = (z-i)(az+b)$ and $f(z) = \dfrac{az+b}{z+i}$

$$\left.\begin{aligned} i &= \lim_{z\to i} \frac{ai+b}{i+i} = \frac{ai+b}{2i} \\ 2 &= \lim_{z\to\infty} \frac{az+b}{z+i} = a \rightsquigarrow a = 2 \end{aligned}\right\} b = -2-2i$$ 6

$\rightsquigarrow$ $p(z) = (z-i)(2z-2-2i) = 2(z-i)(z-1-i)$

$= 2z^2 + (-2-2i-2i)z - i(-2-2i)$

$\underline{= 2z^2 + (2-4i)z + (-2+2i)}$

(d) $\displaystyle f(z) = \frac{p(z)}{z(z-i)}$

$\lim_{z\to 0} f(z)$ exists & z factor of denom $\rightsquigarrow$ z factor of $p(z)$

$\lim_{z\to -i} f(z) = 0$ $\rightsquigarrow$ $(z+i)$ factor of $p(z)$

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$\lim_{z\to 3-i} f(z) = 0$ $\rightsquigarrow$ $(z-3+i)$ factor of $p(z)$

simplest possibility: $p(z) = a z (z+i)(z-3+i)$

$$\rightsquigarrow \quad f(z) = \frac{a(z+i)(z-3+i)}{(z-i)} \qquad \text{with } f(-i) = 0 = f(3-i) = 0 \checkmark$$

$$3i = f(0) = \frac{a\, i\,(-3+i)}{-i} \quad \rightsquigarrow \quad a = \frac{3i}{3-i} = \frac{3i(3+i)}{9+1} = \frac{-3+9i}{10}$$

$$\underline{p(z) = \frac{3}{10}(-1+3i)\, z\, (z+i)(z-3+i)}$$

12) (a) $f(z) = z^2 + i\bar{z} - 1$

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z continuous $\rightsquigarrow$ $z^2 = z\cdot z$ continuous

$\bar{z}$ continuous $\rightsquigarrow$ $i\bar{z}$ continuous

-1 continuous

sum is <u>continuous</u> <u>for all $z \in \mathbb{C}$</u>

(b) $f(z) = i\left(\bar{z}/z\right)^4 \qquad z \in \mathbb{C}\setminus\{0\}$

$f(0) = i$

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<u>continuous for all $z \in \mathbb{C}\setminus\{0\}$</u>, as

i, $\bar{z}$, $1/z$ continuous and products are, too.

$z = 0$: $\qquad z = \varepsilon(1+\alpha i) \qquad \alpha \in \mathbb{R}$

$$\lim_{\varepsilon\to 0} f(\varepsilon(1+\alpha i)) = \lim_{\varepsilon\to 0} i\left[\frac{\cancel{\varepsilon}(1-\alpha i)}{\cancel{\varepsilon}(1+\alpha i)}\right]^4$$

$$= i\left(\frac{1-\alpha i}{1+\alpha i}\right)^4 \quad \text{depends on } \alpha$$

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$\rightsquigarrow$ <u>not continuous at $z=0$</u>. $\quad$ ($f(0)=i$ doesn't matter)

13) (a) $$f'(i) = \lim_{\Delta z\to 0} \frac{(i+\Delta z)^3 - 2(i+\Delta z) - (i^3 - 2i)}{\Delta z}$$

$$= \lim_{\Delta z\to 0} \left(\cancel{i^3} + 3i^2\Delta z + 3i(\Delta z)^2 + (\Delta z)^3 - \cancel{2i} - 2\Delta z \cancel{- i^3} \cancel{+ 2i}\right)/\Delta z$$

$$= \lim_{\Delta z\to 0} \left(-3 + 3i\Delta z + (\Delta z)^2 - 2\right) = -3-2 = -5$$

$\rightsquigarrow$ <u>$f'(i) = -5$</u>

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(b) $$f'(z) = \lim_{\Delta z \to 0} \frac{(z+\Delta z)^2 \cancel{+1} - z^2 \cancel{+1}}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{\cancel{z^2} + 2z\Delta z + (\Delta z)^2 \cancel{-z^2}}{\Delta z} = 2z \qquad 7$$

$$\underline{f'(z) = 2z}$$

(c) $$f'(z_0) = \lim_{\Delta z \to 0} \frac{(z_0+\Delta z)(\bar{z}_0 + \overline{\Delta z}) - 2(z_0+\Delta z) - z_0\bar{z}_0 + 2z_0}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \left(\bar{z}_0 + z_0 \frac{\overline{\Delta z}}{\Delta z} + \overline{\Delta z} - 2 \right)$$

$\lim_{\Delta z \to 0} \overline{\Delta z}/\Delta z$ does not exist $\leadsto$ $\underline{f'(z_0) \text{ does not exist}}$ 6

$\underline{\text{for } z_0 \neq 0}$

for $z_0 = 0$, $\underline{f'(0) = -2}$ 5