MAS205 Complex Variables 2004-2005

Exercises 4

Exercise 14: For each of the following functions $f(x+iy) = u(x,y) + iv(x,y)$, find the set of all points (x,y) at which u and v satisfy the Cauchy-Riemann differential equations $(\partial u/\partial x = \partial v/\partial y$ and $(\partial u/\partial y = -\partial v/\partial x)$.

(a) $f(x+iy) = y^2 + 2ixy$

(b) $f(x+iy) = 2xy - ix + 2iy^3/3$.

Exercise 15: Let $f(z) = e^{z^2/2}$. Write $f(z)$ as $u(x,y)+iv(x,y)$ and show that u and v satisfy the Cauchy-Riemann differential equations. Write

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

and use this to express $f'(z)$ as a function of z.

Exercise 16: Use the Cauchy-Riemann differential equations to find at which values of z the following functions are differentiable. Find the derivative of the functions at these points.

(a) $f(x+iy) = 3x^2y - y^3 + i(3xy^2 - x^3)$

(b) $f(x+iy) = 3x^2y - y^3 + i(x^3 - 3xy^2)$

(c) $f(x+iy) = 2xy^2 + i(x + 2y^3/3)$

(d) $f(z) = (z+\overline{z})(z-\overline{z})^2$.

Exercise 17: Let f and g denote functions $\mathbb{C} \to \mathbb{C}$. For each question below, give either a proof or a counterexample to justify your answer.

(a) If f and g are both differentiable at z_0, does it follow that $g - f$ is continuous at z_0?

(b) If f and g are both non-differentiable at z_0, does it follow that fg is non-differentiable at z_0?

(c) If f and g are both differentiable at z_0, does it follow that $f \circ g$ is differentiable at z_0?

(d) Suppose f is non-differentiable at $3+i$, but differentiable everywhere else, and g is non-differentiable at $2+i$, but differentiable everywhere else. Is $f+g$ differentiable at $4+2i$?

Please hand in your solutions (to the yellow Complex Variables box on the ground floor) by 11am Tuesday 2nd November

Thomas Prellberg, October 2004

25 each

14)

(a) $u(x,y) = y^2$ $\quad \frac{\partial u}{\partial x} = 0 \quad \frac{\partial u}{\partial y} = 2y$

$v(x,y) = 2xy$ $\quad \frac{\partial v}{\partial x} = 2y \quad \frac{\partial v}{\partial y} = 2x$

C.R. $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \rightsquigarrow 2x = 0 \rightsquigarrow x = 0$ 10

$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \rightsquigarrow 2y = -2y \rightsquigarrow y = 0$

satisfied at $z = 0$ $\quad (f'(0) = 0)$

(b) $u(x,y) = 2xy$ $\quad \frac{\partial u}{\partial x} = 2y \quad \frac{\partial u}{\partial y} = 2x$

$v(x,y) = -x + \frac{2}{3}y^3$ $\quad \frac{\partial v}{\partial x} = -1 \quad \frac{\partial v}{\partial y} = 2y^2$

C.R. $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \rightsquigarrow 2y = 2y^2 \rightsquigarrow y = 0$ or $y = 1$

$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \rightsquigarrow 2x = -(-1) \rightsquigarrow x = 1/2$

15

satisfied at $z = 1/2$ $\quad (f'(\tfrac{1}{2}) = 0 - i = -i)$

and $z = 1/2 + i$ $\quad (f'(\tfrac{1}{2} + i) = 2\tfrac{1}{2} - i = 1 - i)$

15) $f(z) = e^{z^2/2}$

$z = x+iy \leadsto e^{z^2/2} = e^{\frac{x^2-y^2}{2} + ixy} = e^{\frac{x^2-y^2}{2}} \cos(xy) + i e^{\frac{x^2-y^2}{2}} \sin(xy)$

$u(x,y) = e^{\frac{x^2-y^2}{2}} \cos(xy)$

$v(x,y) = e^{\frac{x^2-y^2}{2}} \sin(xy)$ 7

$$\frac{\partial u}{\partial x} = e^{\frac{x^2-y^2}{2}} x \cos(xy) + e^{\frac{x^2-y^2}{2}} (-\sin(xy)) y$$

$$= e^{\frac{x^2-y^2}{2}} (x\cos(xy) - y\sin(xy))$$

$$\frac{\partial v}{\partial y} = e^{\frac{x^2-y^2}{2}} (-y)\sin(xy) + e^{\frac{x^2-y^2}{2}} \cos(xy)\, x = \frac{\partial u}{\partial x} \quad ✓$$

$$\frac{\partial u}{\partial y} = e^{\frac{x^2-y^2}{2}} (-y)\cos(xy) + e^{\frac{x^2-y^2}{2}} (-\sin(xy))\, x$$

$$= e^{\frac{x^2-y^2}{2}} (-y\cos(xy) - x\sin(xy))$$

$$\frac{\partial v}{\partial x} = e^{\frac{x^2-y^2}{2}} x\sin(xy) + e^{\frac{x^2-y^2}{2}} \cos(xy)\, y = -\frac{\partial u}{\partial y} \quad ✓ \; ||$$

$$f'(z) = e^{\frac{x^2-y^2}{2}} (x\cos(xy) - y\sin(xy)) + i e^{\frac{x^2-y^2}{2}} (y\cos xy + x\sin(xy))$$

$$= e^{\frac{x^2-y^2}{2}} (x+iy)(\cos xy + i\sin xy)$$

$$= (x+iy)\, e^{\frac{x^2-y^2}{2} + ixy} = (x+iy)\, e^{\frac{(x+iy)^2}{2}}$$

$$= z\, e^{z^2/2}$$ as to be expected. 7

1b) (a)

$$u(x,y) = 3x^2y - y^3 \qquad \frac{\partial u}{\partial x} = 6xy \qquad \frac{\partial u}{\partial y} = 3x^2 - 3y^2$$

$$v(x,y) = 3xy^2 - x^3 \qquad \frac{\partial v}{\partial x} = 3y^2 - 3x^2 \qquad \frac{\partial v}{\partial y} = 6xy$$

CR: $6xy = 6xy$ ✓

hold $\forall z \in \mathbb{C}$

$3x^2 - 3y^2 = -3y^2 + 3x^2$ ✓

f is differentiable forall $z \in \mathbb{C}$, $f'(z) = 6xy + i(3y^2 - 3x^2)$

[$f(z) = -iz^3$, $f'(z) = -3iz^2$ in disguise] 6

(b)

$$u(x,y) = 3x^2y - y^3 \qquad \frac{\partial u}{\partial x} = 6xy \qquad \frac{\partial u}{\partial y} = 3x^2 - 3y^2$$

$$v(x,y) = x^3 - 3xy^2 \qquad \frac{\partial v}{\partial x} = 3x^2 - 3y^2 \qquad \frac{\partial v}{\partial y} = -6xy$$

CR: $6xy = -6xy \rightsquigarrow xy = 0 \rightsquigarrow x = 0$ or $y = 0$

$3x^2 - 3y^2 = -3x^2 + 3y^2 \rightsquigarrow x^2 = y^2 \rightsquigarrow x = y$ or $x = -y$

only satisfied at $z = 0$, $f'(0) = 0$ (part. derivatives are continuous there!)

[$f(z) = i\bar{z}^3$ in disguise] 6

(c)

$$u(x,y) = 2xy^2 \qquad \frac{\partial u}{\partial x} = 2y^2 \qquad \frac{\partial u}{\partial y} = 4xy$$

$$v(x,y) = x + 2y^3/3 \qquad \frac{\partial v}{\partial x} = 1 \qquad \frac{\partial v}{\partial y} = 2y^2$$

CR: $2y^2 = 2y^2$ ✓

$1 = -4xy \rightsquigarrow xy = -\frac{1}{4}$ Hyperbola

$f(z)$ differentiable at

$y = -1/4x$: $z = x - \frac{i}{4x}$ (part. derivatives continuous!)

$$f'(z) = 2\left(-\frac{1}{4x}\right)^2 + i = \frac{1}{8x^2} + i$$ 6

(d) $f(z) = (z+\bar{z})(z-\bar{z})^2 = 2x(2iy)^2 = -8xy^2$

$u = -8xy^2 \qquad \frac{\partial u}{\partial x} = -8y^2 \qquad \frac{\partial u}{\partial y} = -16xy$

$v = 0 \qquad \frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} = 0$

$-8y^2 = 0 \rightsquigarrow y = 0$, $-16xy = 0$ satisfied.

differentiable at $z = x$ (part. derivatives continuous!)

$f'(x) = 0 + i0 = 0$ ✓ 7

(17) (a) yes: g is diff'able, $-f$ is diff'able

$\rightsquigarrow f - g$ is diff'able $\rightsquigarrow f \cdot g$ is continuous 6

(b) no: $f(z) = \bar{z}$, $g(z) = \frac{1}{\bar{z}}$, $f \cdot g(z) = \bar{z}/\bar{z} = 1$

$\rightsquigarrow fg$ is diff'able for all z 6

(c) no: need differentiability of f at $g(z_0)$, not z_0

example: $f(z) = |z|^2$, $g(z) = 1+z$, $z_0 = 0$ 7

$f(z)$, $g(z)$ diff'able at $z_0 = 0$, but $f \circ g(z) = |1+z|^2$ is not

(d) yes: f and g are both differentiable at $4+2i$, 6

so $f+g$ is, as well.