MAS205 Complex Variables 2004-2005

Exercises 5

Exercise 18: Find the radius of convergence of the following power series

$$(a)\quad \sum_{n=1}^{\infty} \frac{z^n}{n^{10}}\,, \quad (b)\quad \sum_{n=0}^{\infty} \frac{z^n}{(3i)^n}\,, \quad (c)\quad \sum_{n=0}^{\infty} z^n \exp(-n)\,, \quad (d)\quad \sum_{n=0}^{\infty} n! z^n\,.$$

Exercise 19: Give an example, if possible, of power series with the following properties:

(a) centred at $z_0 = 4i$, with radius of convergence $R = 4$

(b) centred at $z_0 = 1 + 2i$, with radius of convergence $R = 0$

(c) centred at $z_0 = -2$ and convergent for all z with $\Im(z) < 4$ but divergent for all z with $\Im(z) > 4$

(d) centred at $z_0 = -4i/3$, with radius of convergence $R = \infty$

(e) centred at $z_0 = 0$ and convergent for all z with $\Re(z) = 8$ but divergent for all other $z \in \mathbb{C}$

(Proofs are not necessary, but if you can't find an example you should explain why.)

Exercise 20: Compute the product of the Taylor series of $(1-z)^{-1}$ and $(1+z)^{-1}$ at $z_0 = 0$ and show that the result is equal to the Taylor series of $(1-z^2)^{-1}$ at $z_0 = 0$.

Exercise 21: Let $D = \{z : |z + 3i| < 4\}$. Suppose that $f : D \to \mathbb{C}$ is defined by

$$f(z) = \sum_{n=0}^{\infty} \frac{(z+3i)^n}{(4i)^n}\,.$$

Calculate the Taylor series for f at the point $z_0 = 0$ and determine its radius of convergence.

Exercise 22: Let

$$f(z) = \frac{1}{(z+1)(z-3)}\,.$$

(a) Write down the Laurent series for f on $\{z : |z| > 3\}$.

(b) Write down the Laurent series for f on a punctured disk centred at $z = -1$. For what values of z dues this series converge?

Please hand in your solutions (to the yellow Complex Variables box on the ground floor) by 11am Tuesday 16th November

Thomas Prellberg, November 2004

20 each

18) (a) $\left|\frac{z^{n+1}}{(n+1)^{10}} \Big/ \frac{z^n}{n^{10}}\right| = \left|\frac{z}{(1+\frac{1}{n})^{10}}\right| \xrightarrow[n\to\infty]{} |z|$ 5

$\rightsquigarrow \quad R = 1$

(b) $\left|\frac{z^{n+1}}{(3i)^{n+1}} \Big/ \frac{z^n}{(3i)^n}\right| = \frac{|z|}{3} \quad \rightsquigarrow \quad R = 3$ 5

(c) $\left|\frac{z^{n+1}e^{-(n+1)}}{z^n e^{-n}}\right| = \frac{|z|}{e} \quad \rightsquigarrow \quad R = e$ 5

(d) $\left|\frac{z^{n+1}(n+1)!}{z^n \, n!}\right| = |z| n \xrightarrow[n\to\infty]{} \infty \quad \rightsquigarrow \quad R = 0$

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19) (a) $\sum_{n=0}^{\infty} \left(\frac{z-4i}{4}\right)^n$ 4

(b) $\sum_{n=0}^{\infty} n! \, (z-1-2i)^n$ 4

(c) impossible, domain of convergence needs to be disk 4
(need Dirichlet Series)

d) $\sum_{n=0}^{\infty} \frac{1}{n!}\left(z+\frac{4i}{3}\right)^n$ 4

e) impossible, domain of convergence needs to be disk 4

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4 4

20) $|z|<1$: $$\frac{1}{1-z}\,\frac{1}{1+z} = \sum_{n=0}^{\infty} z^n \sum_{m=0}^{\infty} (-1)^m z^m$$

4 4

$$= \sum_{n,m=0}^{\infty} (-1)^m z^{n+m} = \sum_{N=0}^{\infty} z^N \sum_{\substack{n+m=N \\ n,m\geq 0}} (-1)^n$$

abs. conv.

or similar steps

[the last sum is $\underbrace{1-1+1-1+1\ldots}_{N+1 \text{ terms}} = \begin{cases} 1 & N \text{ even} \\ 0 & N \text{ odd} \end{cases}$]

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$$= \sum_{N=0}^{\infty} z^{2N} = \frac{1}{1-z^2}$$

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21) $$f(z) = \sum_{n=0}^{\infty} \frac{(z+3i)^n}{(4i)^n} \qquad |z+3i|<4$$

$$f(z) = \frac{1}{1-\frac{z+3i}{4i}} \;7\; = \frac{4i}{4i-z-3i} = \frac{4i}{i-z} = \frac{4}{1+iz} \;6$$

$$\leadsto f(z) = \sum_{n=0}^{\infty} 4(-i)^n z^n \;7 \qquad |z|<1$$

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22) $f(z) = \frac{1}{(z+1)(z-3)} = \frac{1}{4}\left(\frac{1}{z-3} - \frac{1}{z+1}\right)$ 3

(a) $|z| > 3$: $f(z) = \frac{1}{4z}\left(\frac{1}{1-3/z} - \frac{1}{1+1/z}\right)$

$$= \frac{1}{4z}\left\{\sum_{n=0}^{\infty}\left(\frac{3}{z}\right)^n - \sum_{n=0}^{\infty}\left(-\frac{1}{z}\right)^n\right\}$$ 3 3

$$= \sum_{n=1}^{\infty}\frac{3^n-(-1)^n}{4z^{n+1}} = \sum_{n=2}^{\infty}\frac{3^{n-1}+(-1)^n}{4}z^{-n}$$ 3

(b) $f(z) = \frac{1}{4}\left(\frac{1}{z+1-4} - \frac{1}{z+1}\right)$

$$= -\frac{1}{4}\frac{1}{z+1} - \frac{1}{16}\frac{1}{1-\frac{z+1}{4}}$$

$$= -\frac{1}{4}\frac{1}{z+1} - \frac{1}{16}\sum_{n=0}^{\infty}\frac{(z+1)^n}{4^n}$$ 3

convergent for $0<|z+1|<4$ 3

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