MAS205 Complex Variables 2004-2005

Exercises 6

Exercise 23: Find the Laurent series of the function

$$f(z) = \frac{1}{(z+3)(z-2)^2}$$

on a punctured disk centred at the point $z_0 = 2$.
Where is this Laurent series valid (i.e. absolutely convergent)?
What is the principal part of this Laurent series?
What type of singularity does f have at $z_0 = 2$?
What is the residue of f at $z_0 = 2$?

Exercise 24: Locate the singularities for each of the following functions, and determine the nature of each singularity:

$$(a)\quad \frac{1}{z^4-16} \qquad (b)\quad \frac{1}{(z-1)^4}+e^{-1/(z+3)} \qquad (c)\quad z^2(e^{1/z^2}-1) \qquad (d)\quad \frac{\sin(z^2)}{z^2}$$

Exercise 25:

(a) List the singularities of the function $f(z) = e^{-iz}/(z^2 - \pi^2)$ and determine the nature of each singularity. Compute the residue of f at each singularity.

(b) List the singularities of the function $f(z) = e^{1/z}/(1+z)$ and determine the nature of each singularity. Compute the residue of f at each singularity.

Exercise 26: Let f and g be holomorphic on a disk D centred at z_0, and let h be holomorphic on the punctured disk $D' = D\backslash\{z_0\}$. Suppose f and g both have zeros of order $m \geq 1$ at z_0 and h has a pole of order $n \geq 1$ at z_0.

(a) Does fh have a zero or pole at z_0? If so, what is its order?

(b) Does $f + g$ have a zero or pole at z_0? If so, what is its order?

Exercise 27: Let E denote the set of all entire functions.

(a) Is E a group under addition (i.e. $(f+g)(z) = f(z) + g(z)$)?

(b) Is E a group under multiplication (i.e. $(fg)(z) = f(z)g(z)$)?

(c) Is E a group under composition (i.e. $(f \circ g)(z) = f(g(z))$)?

Prove your answers.

Note: determining the type of singularity means finding out whether it is a pole (if so, which order?), an essential singularity, or a removable singularity.
Please hand in your solutions (to the yellow Complex Variables box on the ground floor) by 11am Tuesday 23rd November

Thomas Prellberg, November 2004

23)

$$f(z) = \frac{1}{(z+3)(z-2)^2} = \frac{1}{(z-2)^2}\,\frac{1}{5+(z-2)}$$

$$= \frac{1}{5(z-2)^2}\sum_{n=0}^{\infty}\left[-\frac{1}{5}(z-2)\right]^n$$

$$= \sum_{n=0}^{\infty}\frac{(-1)^n}{5^{n+1}}(z-2)^{n-2} = \sum_{n=-2}^{\infty}\frac{(-1)^n}{5^{n+3}}(z-2)^n$$

$$= \frac{1}{5}\,\frac{1}{(z-2)^2} - \frac{1}{25}\,\frac{1}{(z-2)^2} + \sum_{n=0}^{\infty}\frac{(-1)^n}{5^{n+3}}(z-2)^n \qquad 8$$

abs convergent for $0<|z-2|<5$ 3

principal part $\frac{1}{5}\frac{1}{(z-2)^2} - \frac{1}{25}\frac{1}{(z-2)^2}$ 3

pole of order 2 at $z_0 = 2$ 3

residue $-\frac{1}{25}$ 3

24) (a) simple poles at $\pm 2,\ \pm 2i$ 5

(b) pole of order 4 at 1, essential sing. at -3 5

(c) essential singularity at 0 $\left(f(z) = z^2\sum_{n=1}^{\infty}\frac{z^{-2n}}{n!}\right)$ 5

(d) removable singularity at 0 $f(z) = \frac{1}{z^2}\left(z^2 - \frac{z^6}{3!} + \dots\right)$ 5

25) (a) simple poles at $z = \pm\pi$ 4

$$f(z) = \frac{e^{-iz}}{2\pi}\left(\frac{1}{z-\pi} - \frac{1}{z+\pi}\right)$$

residue at π: $+\frac{e^{-i\pi}}{2\pi} = -\frac{1}{2\pi}$ 3

residue at $-\pi$: $-\frac{e^{i\pi}}{2\pi} = +\frac{1}{2\pi}$ 3

(b) simple pole at $z = -1$, essential sing at $z = 0$ 4

$$f(z) = \frac{e^{1/z}}{1+z}$$

residue at -1: $-e^{-1} = -1/e$ 3

residue at 0: $\sum_{n=0}^{\infty} \frac{1}{n!} z^{-n} \sum_{m=0}^{\infty} (-z)^m$ 3

coefficient of z^{-1} is $\frac{1}{1!} - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!} = e^{-1} - 1$

26) $f(z) = F(z)\,(z-z_0)^m$ $F(z_0) \neq 0$, F holo on D 2

$g(z) = G(z)\,(z-z_0)^m$ $G(z_0) \neq 0$, G holo on D 2

$h(z) = H(z)\,(z-z_0)^{-n}$ $H(z_0) \neq 0$, H holo on D' 2

a) $f(z)h(z) = F(z)H(z)\,(z-z_0)^{m-n}$ FH holo on D'

$FH(z_0) \neq 0$ 2

$m>n$: zero of order $m-n$ (removable singularity) 2

$m=n$: neither zero nor pole 2

$m<n$: pole of order $n-m$ 2

b) $f(z)+g(z) = [F(z)+G(z)](z-z_0)^m$ $F+G$ holo on D 2

zero of **at least** order m 2 $F(z_0)$ & $G(z_0)$ **may** be zero 2

27)

(a) yes. 2 assoc ✓ 2

closure ✓ 2

identity $e(z)=0$ ✓ 2

inverse $(-f)(z) = -f(z)$ ✓ 2

(b) no, inverse doesn't exist 2: identity $e(z)=1$

inverse $(f^{-1})(z) = \frac{1}{f(z)}$ not entire 3

(pole at z_0 where $f(z_0)=0$)

(c) no, inverse doesn't exist 2, identity $e(z)=z$

3

problem is lack of injectivity: inverse of e.g. $f(z)=z^2$ not entire.