MAS205 Complex Variables 2004-2005

Exercises 7

Exercise 28: Let the curve $\mathcal{C}$ be given by the graph of the function $y = f(x)$ with

$$f(x) = \frac{x^2}{8} - \log x \qquad (1 \le x \le 2)$$

embedded in $\mathbb{C}$ via $z = x + iy$.

(a) Give a path $\gamma : [a, b] \to \mathbb{C}$ which has the curve $\mathcal{C}$ as its image. Draw a sketch of the curve and indicate the parametrisation.

(b) Compute the length $L(\mathcal{C})$. Evaluate the result numerically and discuss it in view of your sketch (i.e. does your result make sense and why).

Exercise 29: Let $\mathcal{C}$ be the unit circle described counterclockwise. Show that

$$\left| \int_{\mathcal{C}} \frac{\cos z}{z} dz \right| < 2\pi e\ .$$

Exercise 30: Using the definition of the integral of a complex function f along a contour $\gamma : [a, b] \to \mathbb{C}$ as

$$\int_a^b f(\gamma(t))\gamma'(t)dt\ ,$$

compute the integral of $f(z) = (z - 4)^2$ along the straight line segments

(a) from 0 to 2,

(b) from 0 to $-3i$.

Check your answers by finding an antiderivative F for f and evaluating F at the points $z = 0, 3, -2i$.

Exercise 31: Let $f(z) = \bar{z}$. Find the values of $\int_{\mathcal{C}_k} f(z)dz$ where

(a) $\mathcal{C}_1$ denotes the straight line from $z_0 = 2$ to $z_1 = 2i$,

(b) $\mathcal{C}_2$ denotes the arc from $z_0 = 2$ to $z_1 = 2i$ along a circle of radius 2 about the origin.

Find a simple closed contour $\mathcal{C}$ for which $\int_{\mathcal{C}} f(z)dz \ne 0$.

Exercise 32: By applying Cauchy's theorem (or otherwise) show that $\int_{\mathcal{C}} f(z)dz = 0$ where $\mathcal{C}$ is the unit circle $\{z \in \mathbb{C} : |z| = 1\}$ and

$$(a) \quad f(z) = \frac{1}{z^2 + 3} \qquad (b) \quad f(z) = \frac{1}{z^2 + 2iz - 5} \qquad (c) \quad f(z) = \frac{1}{\cosh z}$$

Please hand in your solutions (to the yellow Complex Variables box on the ground floor) by 11am Tuesday 30th November

Thomas Prellberg, November 2004

20 each

28) a) $\gamma(t) = t + i\left(\frac{t^2}{8} - \log t\right)$, $t \in [1,2]$ 3

(can also be different param.)

b) $\gamma'(t) = 1 + i\left(\frac{t}{4} - \frac{1}{t}\right)$ 1

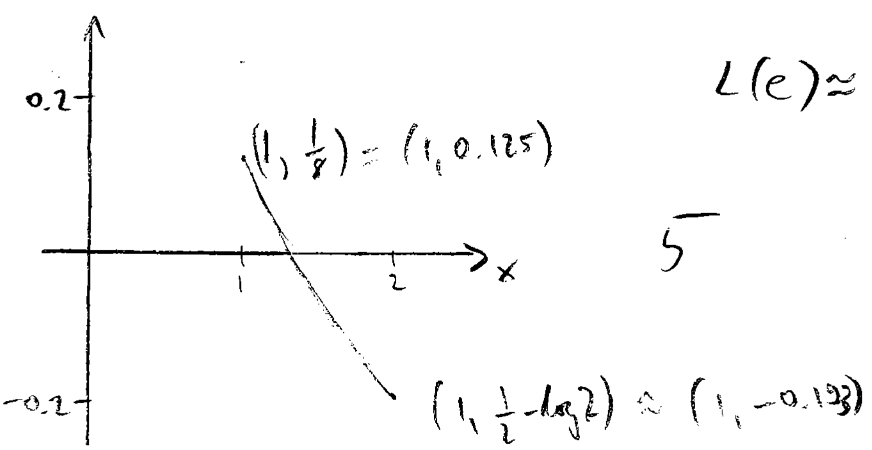
$$|\gamma'(t)|^2 = 1 + \left(\frac{t}{4} - \frac{1}{t}\right)^2 = \frac{\cancel{16}\,8\,t^2 + t^4 - \cancel{8t^2} + 16}{16t^2}$$

$$= \left(\frac{t^2+4}{4t}\right)^2 \rightsquigarrow |\gamma'(t)| = \frac{t^2+4}{4t}$$ 4

$$L(c) = \int_1^2 \frac{t^2+4}{4t}\,dt = \left(\frac{t^2}{8} + \log t\right)\Big|_1^2$$

$$= \frac{4-1}{8} + \log 2 - \log 1 = \frac{3}{8} + \log 2$$ 3

Sketch (not to scale)



$L(c) \approx 1.068$ 1

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length ought to be a bit larger than one ✓

(or: a wee bit larger than $\sqrt{1+(0.125+0.193)^2} \approx 1.05$ ✓)

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2g) $\gamma(t) = e^{it} \quad 0 \le t \le 2\pi \; ; \quad \gamma'(t) = i e^{it}$

$$\left| \int_{\mathcal{C}} \frac{\cos z}{z} \, dz \right|$$

$$= \left| \int_0^{2\pi} \cos(e^{it}) \frac{i \cancel{e^{it}}}{\cancel{e^{it}}} \, dt \right|$$

$$\leq \underbrace{\int_0^{2\pi} dt}_{L = 2\pi} \; \underbrace{\max_{0 \le t \le 2\pi} |\cos(e^{it})|}_{M} \; ; \text{ or } \left| \int \ldots dz \right| \le L M$$

length of contour ↓ L; max of f along contour ↙ M

$\underbrace{}_{4}$ $\quad$ 6

now $\cos e^{it} = \frac{1}{2} \left(e^{ie^{it}} + e^{-ie^{it}} \right)$

$$|\cos e^{it}| \le \frac{1}{2} \left(|e^{ie^{it}}| + |e^{-ie^{it}}| \right)$$

$$= \frac{1}{2} \left(e^{\mathrm{Re}(ie^{it})} + e^{-\mathrm{Re}(ie^{it})} \right)$$

$$= \frac{1}{2} \left(e^{-\sin t} + e^{\sin t} \right) \le e^{|\sin t|} \le e \quad 8$$

$$\rightsquigarrow \left| \int_{\mathcal{C}} \frac{\cos z}{z} \, dz \right| \le 2\pi e \quad ✓$$

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30) $f(z) = (z-4)^2$

(a) $\gamma(t) = 2t \qquad 0 \leq t \leq 1 \qquad \gamma'(t) = 2 \qquad 3$

$$\int_{[0,2]} f(z)\,dz = \int_0^1 (2t-4)^2\, 2\, dt = \frac{8}{3}(t-2)^3 \Big|_0^1 = \frac{8}{3}\left((-1)^3 - (-2)^3\right)$$

$$= \frac{56}{3} \qquad 4$$

(b) $\gamma(t) = -3it \qquad 0 \leq t \leq 1 \qquad \gamma'(t) = -3i \qquad 3$

$$\int_{[0,-3i]} f(z)\,dz = \int_0^1 (-3it-4)^2(-3i)\,dt = \frac{(-3i)^3}{3}\left(t + \frac{4}{3i}\right)^3 \Big|_0^1$$

$$= 9i\left[\left(1 - \frac{4}{3}i\right)^3 - \left(-\frac{4}{3}i\right)^3\right]$$

$$= 9i\left[1 - 4i - \frac{16}{3} - \cancel{\frac{64}{27}i} + \cancel{\frac{64}{27}i}\right]$$

$$= 36 - 39i \qquad 4$$

$F(z) = \frac{(z-4)^3}{3} \rightsquigarrow F'(z) = f(z) \qquad 2$

$F(0) = \frac{(-4)^3}{3} = -\frac{64}{3}$

$F(2) - F(0) = \frac{56}{3}$ ✓ 2

$F(2) = \frac{(-2)^3}{3} = -\frac{8}{3}$

$F(-3i) - F(0) = \frac{44+64}{3} - 39i$ ✓ 2

$F(-3i) = \frac{(-3i-4)^3}{3} = -\frac{1}{3}(4+3i)^3$

$= -\frac{1}{3}\left(4^3 + 3\,4^2\,3i + 3\,4\,3^2 i^2 + 3^3 i^3\right) = \frac{44}{3} - 39i$

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31) $f(z) = \bar{z} = x - iy = re^{-i\varphi}$

(a) $\gamma_1(t) = 2 + 2(i-1)t \quad , \quad 0 \leq t \leq 1 \quad ; \gamma_1'(t) = (i-1)$ 4

$$\int_{C_1} \bar{z}\, dz = \int_0^1 [2 + 2(-i-1)t]\, 2(i-1)\, dt$$

$$= 2(i-1) \left[2t - 2(1+i)\frac{t^2}{2}\right]_0^1 = 2(i-1)\left[2 - 2(1+i)\frac{1}{2}\right]$$

$$= 2(i-1)[1-i] = +4i$$ 4

(b) $\gamma_2(t) = 2e^{it} \quad , \quad 0 \leq t \leq \frac{\pi}{2} \quad ; \gamma_2'(t) = i2e^{it}$ 4

$$\int_{C_2} \bar{z}\, dz = \int_0^{\pi/2} 2\cancel{e^{-it}}\, i2\cancel{e^{it}}\, dt = 2\pi i$$ 4

Let $C = C_1 - C_2$. Then $\int_C \bar{z}\, dz = (4 - 2\pi)i \neq 0$ 4

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32) (a) f holomorphic on and inside $\mathcal{C}$

(singularities at $\pm\sqrt{3}\,i$ outside $\mathcal{C}$) 6

(b) f holomorphic on and inside $\mathcal{C}$

(singularities at $z_{1,2} = -i \mp \sqrt{-1+5} = \mp 2 - i$ outside $\mathcal{C}$)

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(c) f holomorphic on and inside $\mathcal{C}$

(singularities at $z_k = (2k+1)\frac{\pi}{2}i$, $k \in \mathbb{Z}$ outside $\mathcal{C}$)

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