MAS205 Complex Variables 2004-2005

Exercises 8

Exercise 33: Use the Cauchy integral formula to evaluate each of the following integrals, where C is the circle $\{z \in \mathbb{C} : |z-4| = 2\}$ traversed in the positive (anticlockwise) sense:

(a)
$$\int_C \frac{(z+2)^3}{(z-4)z^2} dz$$

(b)
$$\int_C \frac{4}{(z-\pi)\sin(z/2)} dz$$

Exercise 34: Use the residue theorem to calculate

$$\int_C \frac{1}{(z^2-9)(z+2i)} dz$$

for

(a) $C = C_1$, the positively oriented circle of radius 5 centred at 1;

(b) $C = C_2$, the positively oriented square with corners $-1-3i$ and 2.

Exercise 35: Let $f : \mathbb{C} \to \mathbb{C}$ be holomorphic apart from a simple pole at $3i$ with residue $2\pi i$ and an essential singularity at $2-i$ with residue $3+i$. How many elements are there in the set

$$\left\{ \int_C f(z)dz : C \text{ is a simple closed curve in } \mathbb{C} \setminus \{3i, 2-i\} \right\} ?$$

Exercise 36: Suppose $f : \mathbb{C} \to \mathbb{C}$ is holomorphic except for singularities at the points $-2i$, 1, and $2i$. Let C_1 be the positively oriented circle of radius 3 centred at 0. Let C_2 be the positively oriented circle of radius 2 centred at i. Let C_3 be the positively oriented circle of radius 3 centred at 3. Suppose $\int_{C_1} f(z)dz = 1$, $\int_{C_2} f(z)dz = 2$, and $\int_{C_3} f(z)dz = 3$.

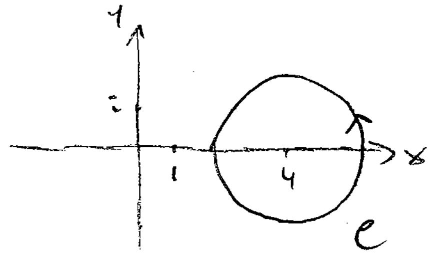
(a) Calculate the residues of f at each of its three singularities.

(b) Give an explicit example of a function f satisfying the above properties.

Please hand in your solutions (to the yellow Complex Variables box on the ground floor) by 11am Tuesday 7th December

Thomas Prellberg, November 2004

33) (25 each)



(a) $$\int_{\mathcal{C}} \frac{(z+2)^2}{(z-4)z^2}\,dz = \int_{\mathcal{C}} \frac{\phi(z)}{z-4}\,dz \quad \left(\text{with } \phi(z) = \frac{(z+2)^2}{z^2}\right.$$

holomorphic on and inside $\left.\mathcal{C}\right) = 2\pi i\,\phi(4) = 2\pi i\,\frac{(4+2)^2}{4^2} = \frac{9}{2}\pi i$

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(b) $$\int_{\mathcal{C}} \frac{4}{(z-\pi)\sin z/2}\,dz = \int_{\mathcal{C}} \frac{\phi(z)}{z-\pi}\,dz = 2\pi i\,\frac{4}{\sin\frac{\pi}{2}} = 8\pi i$$

as $\phi(z) = \frac{4}{\sin z/2}$ is holomorphic on and inside $\mathcal{C}$ (poles at $2k\pi$ are outside)

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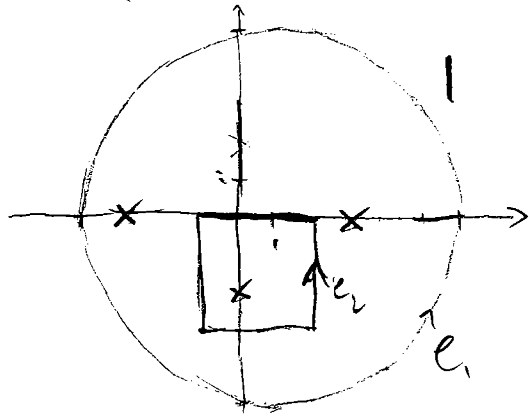
34) $f(z) = \frac{1}{(z^2-9)(z+2i)}$ has simple poles at $\pm 3, -2i$ 4



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$$\operatorname{res}_{-3} f = \frac{1}{(-3-3)(-3+2i)} = \frac{1}{6(3-2i)}$$ 4

$$\operatorname{res}_{3} f = \frac{1}{(3+3)(3+2i)} = \frac{1}{6(3+2i)}$$ 4

$$\operatorname{res}_{-2i} f = \frac{1}{(-2i)^2-9} = -\frac{1}{13}$$ 4

(a) $$\int_{\mathcal{C}_1} f(z)\,dz = 2\pi i\left(\operatorname{res}_{-3} f + \operatorname{res}_{3} f + \operatorname{res}_{-2i} f\right) = 0$$ 4

(b) $$\int_{\mathcal{C}_2} f(z)\,dz = 2\pi i\left(\operatorname{res}_{-2i} f\right) = -\frac{2}{13}\pi i$$ 4

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35) C encircles no sing: 0 ~~3~~ 3

C encircles both sing anti-clockwise $2\pi i\,(2\pi i + 3 + i)$ 3

" clockwise $-2\pi i\,(2\pi i + 3 + i)$ 3

C encircles sing at $3i$, but not at $2-i$

anti-clockwise $2\pi i\,(2\pi i)$ 3

clockwise $-2\pi i\,(2\pi i)$ 3

C encircles sing at $2-i$, but not at $3i$

anticlockwise $2\pi i\,(3+i)$ 3

clockwise $-2\pi i\,(3+i)$ 3

$\to$ ~~7~~ 4 elements in set (if clockwise omitted, 4 elements)

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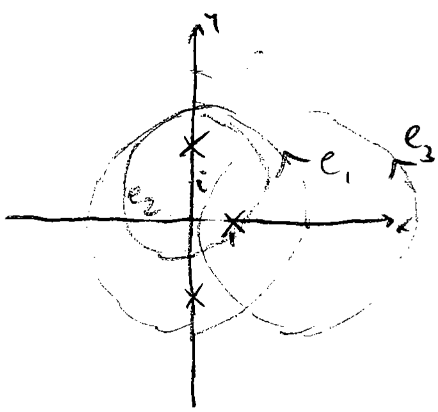
36)



$$1 = \int_{C_1} f(z)\,dz = 2\pi i\,(\mathrm{res}_1 f + \mathrm{res}_{2i} f + \mathrm{res}_{-2i} f)$$ 4

$$2 = \int_{C_2} f(z)\,dz = 2\pi i\,(\mathrm{res}_{2i} f + \mathrm{res}_1 f)$$ 4

$$3 = \int_{C_3} f(z)\,dz = 2\pi i\,(\mathrm{res}_1 f)$$ 4

(a) $\mathrm{res}_1 f = \frac{3}{2\pi i}$ 2, $\mathrm{res}_{2i} f = -\frac{1}{2\pi i}$ 2, $\mathrm{res}_{-2i} f = -\frac{1}{2\pi i}$ 2

(b) $f(z) = -\frac{1}{2\pi i}\,\frac{1}{z+2i} - \frac{1}{2\pi i}\,\frac{1}{z-2i} + \frac{3}{2\pi i}\,\frac{1}{z-3}$

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