MAS205 Complex Variables 2004-2005

Midterm Test, 8th November 2004, 11.05-11.55am

You should attempt ALL questions. Make sure your name and student number is on EVERY sheet handed in. This is a "closed book" test. Calculators are not allowed.

Question 1: [15 marks]

(a) Find all solutions $z \in \mathbb{C}$ of the equation

$$z^3 = -8i \ .$$

(b) Find all solutions $z \in \mathbb{C}$ of the equation

$$e^{-z} = 1 \ .$$

Express all solutions in standard and polar form, and draw diagrams showing their location in the complex plane.

Question 2: [15 marks]
Consider the transformation

$$z \mapsto w = iz^2 \ .$$

(a) Find the equation of the image of the line $\Im(z) = 1$ and sketch the image.

(b) What is the image of the left half plane $\{z \in \mathbb{C} : \Re(z) < 0\}$?

Question 3: [15 marks]
Find the Möbius transformation $f(z) = (az + b)/(cz + d)$ which maps $0 \mapsto i$, $1 \mapsto 0$, and $-1 \mapsto \infty$.

Question 4: [15 marks]
Evaluate

$$\text{(a)} \quad \lim_{z \to 2i} \frac{z^2 - 5iz - 6}{z^2 + 4} \qquad \text{(b)} \quad \lim_{z \to \infty} \frac{(1 - 2z)(1 + 2z)}{1 + iz^2}$$

Question 5: [10 marks]
Show that $\lim_{z \to 0} (\overline{z} - z)^2 / z$ exists.

Question 6: [15 marks]
At what values of $z = x + iy$ is the function $f(x + iy) = x^2 + y^2 - 2xyi$ differentiable?

Question 7: [15 marks]
Let $f(z) = (1 - z)/(1 + z)$. Determine the Taylor series $\sum_{n=0}^{\infty} a_n z^n$ for f around the point $z_0 = 0$. What is the radius of convergence of this Taylor series?

Thomas Prellberg, October 2004

1) (a) $z^3 = -8i = 2^3 e^{i\frac{3\pi}{2}}$

$\rightsquigarrow \quad z_k = 2e^{i\left(\frac{2k}{3}+\frac{1}{2}\right)\pi} \qquad k = 0,1,2$

$z_0 = 2e^{i\frac{\pi}{2}} = 2i$ 1 1

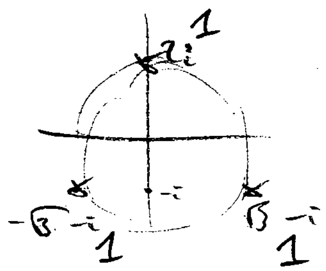
$z_1 = 2e^{7/6\pi i} = -\sqrt{3}-i$ 1 1

$z_2 = 2e^{11/6\pi i} = \sqrt{3}-i$ 1 1 (a): 9

(Check: $(\mp\sqrt{3}-i)^3$

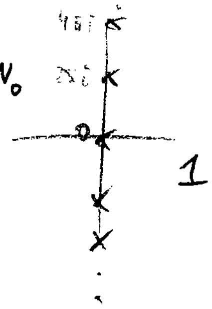
$= \mp3\sqrt{3} - 3\cdot3i \pm 3\sqrt{3} + i$

$= -8i$ ✓)



(b) $e^{-z} = e^0 \to z_k = 2k\pi i$ 3 $k \in \mathbb{Z}$ 1

or $z_k = 2k\pi\, e^{\pm i\frac{\pi}{2}}$, $k \in \mathbb{N}_0$ 1 (b): 6



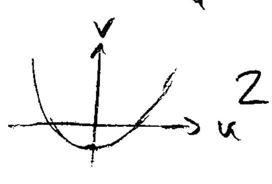
15

2) $z \to w = iz^2$

(a) $\mathrm{Im}(z) = 1 \rightsquigarrow z = x+i$ 1

$w = i(x+i)^2 = i(x^2-1) - 2x$ 1 (a): 8

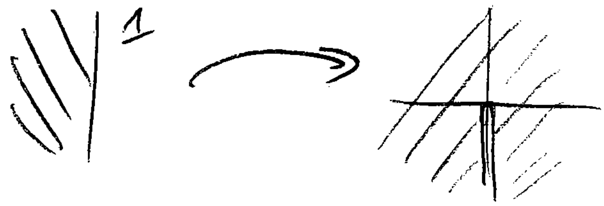
$w = u+iv \rightsquigarrow u = -2x$ 1 $v = x^2-1$ 1 $\rightsquigarrow v = \frac{1}{4}u^2 - 1$ 2



(b) $\operatorname{Re}(z)=0$ gets mapped to neg imag axis

$$re^{i\theta} \rightsquigarrow r^2 e^{i(2\theta+\pi/2)} \quad \tfrac{3}{2}\pi < 2\theta+\tfrac{\pi}{2} < \tfrac{7}{2}\pi \quad 2$$

Image of $\{\operatorname{Re}(z)<0\}$ is $\mathbb{C} \setminus i[0,\infty)$ 2



1 2 (b) 7

15

3) $f(z) = \dfrac{az+b}{cz+d}$

$0 \to i \rightsquigarrow b = id$ 3

$1 \to 0 \rightsquigarrow a = -b$ 3

$-1 \to \infty \rightsquigarrow c = d$ 3

$$f(z) = -i\,\frac{z-1}{z+1} \quad 6$$

check: $f(0) = -i\,\dfrac{0-1}{0+1} = i$ ✓

$f(1) = -i\,\dfrac{1-1}{1+1} = 0$ ✓ 15

$f(-1) = -i\,\dfrac{-1-1}{-1+1} = \infty$ ✓

4) (a) $\displaystyle\lim_{z\to 2i} \frac{z^2-5iz-6}{z^2+4} = \lim_{z\to 2i} \frac{2z-5i}{2z} = \frac{4i-5i}{4i} = -\frac{1}{4}$

4 ✓ (a) 8

or $\dfrac{(z-3i)\cancel{(z-2i)}}{(z+2i)\cancel{(z-2i)}}$

as $(2i)^2+4 = -4+4=0$ 2 and $(2i)^2-5i(2i)-6$ 2

$= -4+10-6=0$

(b) $\displaystyle\lim_{z\to\infty} \frac{(1-2z)(1+2z)}{1+iz^2} = \frac{(-2)(2)}{i} = 4i$ (b) 7

15

5) $\lim_{z\to 0} (\bar z - z)^2 / z$

$= \lim_{z\to 0} \left(\bar z^2 / z - 2\bar z + z \right)$ 2

$/10$

$= \lim_{z\to 0} \cancel{\bar z}\, \bar z / z - \cancel{2 \lim_{z\to 0} \bar z} + \cancel{\lim_{z\to 0} z} = 0$ ✓ 4 2 2

eg: $\frac{\bar z^2}{z} = \frac{r^2 e^{-2i\theta}}{r e^{i\theta}} = r e^{-3i\theta} \xrightarrow[r\to 0]{} 0$

6) $u = x^2 + y^2$ 1 $v = -2xy$ 1 cont diff'able. ! 3

$\frac{\partial u}{\partial x} = 2x$ 1 $= \frac{\partial v}{\partial y} = -2x$ 1 $\rightsquigarrow$ $x = 0$ 1

CR: 2

$\frac{\partial u}{\partial y} = 2y$ 1 $= -\frac{\partial v}{\partial x} = 2y$ 1 ✓ 1

$\rightsquigarrow$ $f(z)$ is differentiable at $\underline{z = iy}$ 2

$/15$

7) $f(z) = \frac{1-z}{1+z} = \frac{1+z-2z}{1+z} = 1 - 2z \sum_{n=0}^{\infty} (-z)^n$

$= 1 + 2 \sum_{n=1}^{\infty} (-1)^n z^n$ 7 $|z| < 1$ 4

$R = 1$ 4

$/15$