1

Noisy dynamical systems with time delay: some basic analytical perturbation schemes with applications

W. Just[1,2], P.M. Geffert[1,2], A. Zakharova[2], and E. Schöll[2]

[1] Queen Mary, University of London, School of Mathematical Sciences, Mile End Road, London E1 4NS, UK `w.just@qmul.ac.uk`
[2] Institut für Theoretische Physik, Technische Universität Berlin, Hardenbergstr. 36, D-10623 Berlin, Germany

Abstract. Systems with time delay, a rather prominent branch in applied dynamical systems theory, constitute a special case of functional differential equations for which the general mathematical theory is fairly well developed and largely parallels the theory of ordinary differential equations. Hence analytic concepts like bifurcation theory, adiabatic elimination, global attractors, invariant manifolds and others can be used to study dynamical behaviour of systems with time delay if some care is applied to take special features of infinite dimensional phase spaces into account. Simple analytic perturbation schemes, frequently used to gain insight for ordinary differential equations, can be applied to time delay dynamics as well. However, such approaches seem to be used infrequently within the physics community, probably because of a lack of easily accessible expositions. Here we review some elementary and well established concepts for the analytical treatment of time delay dynamics, even when subjected to noise. We cover normal form reduction and adiabatic elimination, stochastic linearisation of time delay dynamics with noise, and some elements of weakly nonlinear- and bifurcation analysis. These tools will be illustrated with applications in control problems, time delay autosynchronisation, coherence resonance, and the computation and structure of power spectra in noisy time delay systems.

1.1 Introduction

At about half a century ago the notion of chaos and the importance of nonlinear properties has become relevant in quite diverse branches of science, with the effect that the well developed specialised mathematical discipline of dynamical systems theory became one of the major subjects in applied sciences. Since then, dynamical systems theory has been enriched and diversified by the incorporation of other, important aspects. Noise and imperfections are certainly one of the major issues appearing in applications. The emphasis of this aspect can be traced back for about a century [1–3]. The constructive role of noise becomes apparent when nonlinear dynamical behaviour is considered (see e.g. [4,5]), and this aspect plays a major role for the theory of systems far from equilibrium which is still under development (see e.g. [6]). The various facets of rigorous approaches and a functional analytic setting are rather well developed, with even quite accessible expositions being available [7,8] at the expense of some rigour. In addition, stochastic dynamics has been cast into the language of modern dynamical systems theory, using tools like invariant manifolds and bifurcations [9].

The inclusion of time-delayed interactions and the investigation of the corresponding non-Markovian dynamics has as well a fairly long tradition, mainly triggered by topics in engineering (see e.g. [10] and references therein) or by linear response theories in solid state physics [11, 12]. The proper inclusion of nonlinear dynamical aspects has added additional stimulus to this branch of dynamical systems theory [13–17], in particular, for applications in fast optical systems and communication networks where the speed of light becomes a dynamically relevant quantity [18]. Rigorous mathematical approaches for time delay systems are fairly well developed within a suitable functional theoretical setting [19], underpinning the numerical approaches which largely parallel developments in finite dimensional systems, like, e.g., numerical continuation for uncovering bifurcation structures in time delay dynamics [20].

Here we want to consider analytical approaches for dynamical systems which may have both of the previously mentioned facets in conjunction, i.e., nonlinear stochastic time delay systems. We will focus on simple analytical perturbation schemes and therefore sacrifice to a large extent any mathematical rigour. Instead we focus on formal expansion schemes, which have been well developed and which are in fact quite well known in the context of ordinary differential equations. What we present here is not completely new and can be found, at least indirectly, in the existing specialised literature. However, we have the impression that such approaches are probably not very well known within an applied dynamical systems community. As a consequence of our strategy we will largely skip recent topical developments in time delay dynamics, as these topics normally still defy any systematic analytical treatment. For instance, the general theory of systems with distributed and time dependent delays is well developed and can be dealt with by numerical tools, but analytical results in closed form are rare. Most challenging are systems with state dependent delays were even the fundamental theory is still not completely developed. As for synchronisation in coupled time delay systems master stability function techniques [21] can be applied in the case of uniform time delay and nice results, e.g., in the limit of large delay times can be deduced analytically [22,23], emphasising the role of strong and weak instabilities. However, the power of this approach to disentangle the topology of the coupling from the properties of the single site dynamics disappears when more complicated delay structures are considered. For these advanced cases one still relies largely on numerical simulations with little input from systematic analytical expansions. Similar caveats apply if one intends to generalise Fokker-Planck techniques to noisy time delay dynamics. While it is still possible to derive an equation of motion for the distribution function, the approach suffers from the shortcoming that no closed simple equation of motion can be obtained. Nevertheless, such ideas can be used to study linear dynamical systems [24] and the limit of small delay and thus can quantitatively capture corrections to the Markovian limit [25].

The aim of our contribution is modest. We want to summarise some elementary formally consistent analytic perturbation tools for the analysis of delay dynamics with and without noise, which are valid beyond the limit of small time delay. To some extent our approach can be illustrated by the intuitive averaging technique. Averaging techniques, multiple scaling expansions, and weakly nonlinear analysis all describe similar approaches to deal with oscillatory systems which are in some sense close to a linear case. These concepts have the appeal of an immediate intuitive interpretation, they are quite flexible, they have appeared in various contexts, such as nonlinear oscillators (see, e.g., [26] for an introduction) or coherent pattern formation [27], and can be easily generalised to capture time delay dynamics as well (see, e.g., [28, 29] for some examples in time-delayed feedback control). Multiple scaling techniques are closely related and often largely identical to adiabatic elimination schemes [30], which at the rigorous level can be formalised in terms of centre manifold and normal form reductions. For deterministic time delay dynamics the entire setup is very well established, but not easy to apply by non specialists as one has to master a fairly heavy technical framework which is involved in a rigours description [19]. Some of the more traditional and less general literature is probably easier to access for non-mathematicians [10, 31] and it is this kind of approach we intend to pursue here.

The analytic expansion schemes which are at focus of our interest will largely be based on a thorough understanding of linear time delay dynamics. Even though linear systems are straightforward to solve, either in the deterministic or in the stochastic case, we will summarise the essential features in section 1.2. For those readers familiar with expansion schemes in dynamical systems theory it will not come as a surprise that the proper understanding of the adjoint equation will turn out to be the key for analytic perturbation schemes. As one by-product of our discussion we will describe as well the essential structure of an adiabatic elimination scheme, i.e., the centre manifold reduction in weakly nonlinear noisy time delay systems. We will use two particular examples to illustrate a few aspects of the analytic expansion schemes. Section 1.3 is devoted to time-delayed feedback control, a simple setup for the study of deterministic time delay dynamics. We will use this case to discuss in some detail the analytic investigation of instabilities in time delay systems. The results will show a few, probably general, features shared by time-delayed feedback control systems. As an example for stochastic time delay dynamics we will analyse in section 1.4 a simple model for coherence resonance subjected to time-delayed feedback. We will use this case to illustrate centre manifold reduction and adiabatic elimination in a noisy time delay system, to finally compute the stationary distribution. Correlation functions will be dealt with by stochastic linearisation, to demonstrate the benefit of mean-field methods in a weakly nonlinear setting.

1.2 Some basic features of linear and weakly nonlinear systems

Analytic perturbation schemes for dynamical systems rely to a large extent on the detailed analysis of linear systems. Let us recall some of those ideas in the context of time delay dynamics. All these issues can be found in the literature with various degrees of rigour, and here we largely follow the approach used in [32]. We will focus on an elementary formally consistent approach which does not require any sophisticated mathematical background.

1.2.1 Linear equations and eigenvalue decompositions

Consider a linear inhomogeneous delay differential equation with a single time delay τ

$$\underline{\dot{x}} = \underline{\underline{A}}\,\underline{x} + \underline{\underline{B}}\,\underline{x}(t-\tau) + \underline{h}(t) \tag{1.1}$$

where we allow for a vector valued variable $\underline{x} \in \mathbb{R}^n$ with the coefficients being given by square matrices $\underline{\underline{A}}$ and $\underline{\underline{B}}$. The inhomogeneity $\underline{h}(t)$ can be either a deterministic drive or a stochastic forcing. In fact, with a slight abuse of notation we mean by linear equation a linear equation with constant coefficients. The case of time dependent coefficients would require a completely different approach and would hardly allow for explicit solutions in closed form.

As usual, the corresponding homogeneous system can be reduced to a nonlinear eigenvalue problem

$$\Lambda \underline{u}_\Lambda = \left(\underline{\underline{A}} + \exp(-\Lambda\tau)\underline{\underline{B}}\right) \underline{u}_\Lambda \tag{1.2}$$

when using a particular solutions of exponential type, $\underline{x}(t) = \exp(\Lambda t)\underline{u}_\Lambda$. The eigenvalues are determined by the quasipolynomial

$$\det\left(\underline{\underline{A}} - \Lambda \underline{\underline{I}} + \exp(-\Lambda\tau)\underline{\underline{B}}\right) = 0 \tag{1.3}$$

where $\underline{\underline{I}}$ denotes the identity matrix. Such a transcendental characteristic equation is not straightforward to solve (see, e.g., [10,19,33] for some approaches) as it normally has an infinite set of complex valued solutions. It is however fairly straightforward to see that the real parts of the eigenvalues cannot become arbitrarily large. If that would be the case then the exponential contribution $\exp(-\Lambda\tau)$ in eq.(1.3) tends to zero and the eigenvalues would need to tend towards the finite eigenvalues of $\underline{\underline{A}}$. A similar argument shows that eigenvalues with large imaginary part necessarily have their real parts tending to minus infinity. Hence, even though there are infinitely many eigenmodes almost all are exponentially damped, meaning that the delay dynamics becomes essentially finite dimensional. In eq.(1.1) we have considered delays appearing in the state variable $\underline{x}(t)$ only, the so called "retarded" case. Let us add a remark on time delays appearing in the derivative. If we replace the second term in eq (1.1), say, by $\underline{\underline{B}}\,\underline{\dot{x}}(t-\tau)$ then the corresponding quasipolynomial (1.3) contains a term $\Lambda \exp(-\Lambda\tau)\underline{\underline{B}}$. If we now apply the previous reasoning then it is easily possible to have eigenvalues with large imaginary parts and finite negative real parts by keeping $\Lambda \exp(-\Lambda\tau)$ finite. Hence, for such types of equations we have a huge number of modes which are weakly damped, a case which reminds us of properties of Hamiltonian systems. That is one of the reasons why systems with the delay appearing in the highest derivative behave differently and have been somehow misleadingly termed as "neutral" delay systems. Here we entirely focus on the retarded case.

For solving the original problem, eq.(1.1), i.e., to perform a decomposition in eigenmodes the adjoint equation plays a central role. The adjoint dynamics is in fact a dynamical system running backwards in time, and it seems to be quite challenging to find elementary expositions for time dependent cases in the literature [31]. Here we will just need the adjoint eigenvalue equation which reads

$$\underline{v}_\Lambda^\dagger \Lambda = \underline{v}_\Lambda^\dagger \left(\underline{\underline{A}} + \exp(-\Lambda\tau)\underline{\underline{B}}\right) . \tag{1.4}$$

The eigenvectors obey a kind of "pseudo orthogonality" in the sense that for $\Lambda \neq \Lambda'$

$$\begin{aligned}&\underline{v}^{\dagger}_{\Lambda'}\underline{u}_{\Lambda} + \int_{-\tau}^{0} \exp(-\Lambda'(\theta+\tau))\exp(\Lambda\theta)\underline{v}^{\dagger}_{\Lambda'}\underline{\underline{B}}\,\underline{u}_{\Lambda}d\theta \\ &= \underline{v}^{\dagger}_{\Lambda'}\frac{\Lambda-\Lambda'+(\exp(-\Lambda'\tau)-\exp(-\Lambda\tau))\,\underline{\underline{B}}}{\Lambda-\Lambda'}\underline{u}_{\Lambda} = 0\,. \end{aligned} \tag{1.5}$$

For the last step we used the eigenvalue equations (1.2) and (1.4). We are now able to decompose the inhomogeneous system into eigenmodes if we introduce an appropriate bilinear form which is inspired by eq.(1.5)

$$(V_\Lambda|X_t) = \underline{v}^{\dagger}_{\Lambda}\underline{x}(t) + \int_{-\tau}^{0} \exp(-\Lambda(\theta+\tau))\underline{v}^{\dagger}_{\Lambda}\underline{\underline{B}}\,\underline{x}(t+\theta)d\theta\,. \tag{1.6}$$

One can consider eq.(1.6) to be just a useful abbreviation. Naively, it could be viewed as a kind of scalar product, even though one has to keep in mind that the underlying phase space is not a Hilbert space. By straightforward differentiation of eq.(1.6) one easily verifies that eq.(1.1) reduces to

$$\frac{d}{dt}(V_\Lambda|X_t) = \Lambda\,(V_\Lambda|X_t) + \underline{v}^{\dagger}_{\Lambda}\underline{h}(t) \tag{1.7}$$

i.e., we have decomposed the original problem into a set of decoupled scalar equations. It is possible to reconstruct the solution $\underline{x}(t)$ given the expressions (1.6). However, in the linear case it is often simpler to use alternative methods to compute solutions, e.g., by Laplace or Fourier transforms.

1.2.2 Adiabatic elimination and centre manifold reduction

Dynamical systems with slow modes allow for the adiabatic elimination of fast degrees of freedom. Such a case occurs for instance in the neighbourhood of time independent states when almost all eigenvalues of the linear part are negative and just a few eigenvalues have small positive or vanishing real part. Within the context of time delay dynamics let us assume that by an appropriate coordinate transformation the time independent state is the trivial solution $\underline{x}=0$, and that the linear part of the equations of motion is given by eq.(1.1), i.e.,

$$\dot{\underline{x}}(t) = \underline{\underline{A}}\,\underline{x}(t) + \underline{\underline{B}}\,\underline{x}(t-\tau) + \underline{f}(\underline{x}(t),\underline{x}(t-\tau)) + \underline{\underline{g}}(\underline{x}(t),\underline{x}(t-\tau))\underline{\xi}(t)\,. \tag{1.8}$$

Here $\underline{f}$ denotes the higher order nonlinear terms, and we allow as well for the inclusion of a Gaussian white noise with correlation function $\langle\underline{\xi}(t)\underline{\xi}^T(t')\rangle = \delta(t-t')\underline{\underline{I}}$. We assume that all eigenvalues of the linear equation have negative real part apart from a complex conjugate purely imaginary pair $\pm i\Omega$. The corresponding two eigenfunctions constitute the slow modes, and we are able to reduce the original system (1.8) effectively to the dynamics on this two-dimensional manifold. In order to keep the notation as simple as possible and to avoid the functional analytic description of the phase space let us just recall that the integration of a delay differential equation requires the knowledge of the solution at times $t+\theta$ with $\theta \in [-\tau,0)$ denoting the history. Hence, on our slow manifold we express the solution in the form

$$\underline{x}(t+\theta) = C(t)\exp(i\Omega\theta)\underline{u}_{i\Omega} + \bar{C}(t)\exp(-i\Omega\theta)\bar{\underline{u}}_{i\Omega} + \underline{R}(C(t),\theta) \tag{1.9}$$

where $\exp(i\Omega\theta)\underline{u}_{i\Omega}$ is the relevant slow eigenmode, $C(t)\in\mathbb{C}$ denotes the coordinate on the slow manifold, $\bar{C}(t)$ its complex conjugate, and $\underline{R}$ abbreviates the terms of higher than first order in C. In geometric terms the higher order contributions take care of the deviation of the centre manifold from the linear eigenspace, i.e., $\underline{R}$ takes the curvature of the centre manifold into account. The splitting given by eq.(1.9) has some ambiguity and we have some flexibility to define the higher order terms. We use this freedom to require that the higher order terms are "orthogonal" to the linear eigenspace in the sense of the bilinear form, eq.(1.6). Hence, the condition $(V_{i\Omega}|R) = 0$ results in

$$\begin{aligned}(V_{i\Omega}|X_t) &= C(t)\left(\underline{v}^\dagger_{i\Omega}\underline{u}_{i\Omega} + \int_{-\tau}^{0} \exp(i\Omega\theta)\exp(-i\Omega(\theta+\tau))\underline{v}^\dagger_{i\Omega}\underline{\underline{B}}\,\underline{u}_{i\Omega}d\theta\right)\\ &\quad+\bar{C}(t)\left(\underline{v}^\dagger_{i\Omega}\underline{\bar{u}}_{i\Omega} + \int_{-\tau}^{0} \exp(-i\Omega\theta)\exp(-i\Omega(\theta+\tau))\underline{v}^\dagger_{i\Omega}\underline{\underline{B}}\,\underline{\bar{u}}_{i\Omega}d\theta\right)\\ &= C(t)\left(\underline{v}^\dagger_{i\Omega}\underline{u}_{i\Omega} + \tau\exp(-i\Omega\tau)\underline{v}^\dagger_{i\Omega}\underline{\underline{B}}\,\underline{u}_{i\Omega}\right) . \end{aligned} \tag{1.10}$$

The contribution containing the complex conjugate amplitude $\bar{C}(t)$ vanishes because of the orthogonality (1.5). Since the abbreviation, eq.(1.6) or eq.(1.10), obeys the equation of motion (1.7) when we identify the inhomogeneous part $\underline{h}$ with the nonlinear and stochastic contributions in eq.(1.8), we arrive at

$$\dot{C}(t) = i\Omega C(t) + \frac{\underline{v}^\dagger_{i\Omega}\underline{f}(\underline{x}(t),\underline{x}(t-\tau)) + \underline{v}^\dagger_{i\Omega}\underline{\underline{g}}(\underline{x}(t),\underline{x}(t-\tau))\underline{\xi}(t)}{\underline{v}^\dagger_{i\Omega}\underline{u}_{i\Omega} + \tau\exp(-i\Omega\tau)\underline{v}^\dagger_{i\Omega}\underline{\underline{B}}\,\underline{u}_{i\Omega}} . \tag{1.11}$$

If we take into account that we consider solutions on our slow manifold, i.e., $\underline{x}$ obeys eq.(1.9), we obtain a closed ordinary stochastic differential equation for the amplitude C. The impact of the time delay is contained in the coefficients via the spectrum and the eigenvectors. If our original system has only cubic nonlinearities, a case which for simplicity is often considered in applications, see e.g. [34, 35], then the lowest order linear approximation in eq.(1.9) is sufficient. Otherwise, one would need to compute the nonlinear corrections $\underline{R}$ to the invariant manifold using dynamical invariance. The details are essentially identical to the procedure used for ordinary differential equations. In addition, some greater care is needed when one deals with stochastic dynamics. In fact, the underlying mathematical considerations may become quite involved (see, e.g., [9] for an introduction in the case of stochastic differential equations).

The goal to arrive at an effective equation of motion can be achieved by different, largely equivalent, methods. Multiple scaling techniques (e.g. [26, 36]) are rather efficient to derive equations of motion on slow time scales, even though one needs to identify first a small expansion parameter. To some extent these methods inherently involve a time scale argument which eliminates as well nonresonant terms from equations of motion and thus perform both, a centre manifold and a normal form reduction in one step. Above all, the approaches sketched here rely on a detailed understanding of linear equations of motion, which to a large extent is the backbone of most analytical perturbation schemes.

1.2.3 Linear stochastic equations and power spectral density

An important example of a linear driven system is a stochastic system where the driving force is given by white noise

$$\underline{h}(t) = \underline{\underline{g}}\,\underline{\xi}(t), \qquad \langle \underline{\xi}(t)\underline{\xi}^T(t')\rangle = \delta(t-t')\underline{\underline{I}} \tag{1.12}$$

with $\underline{\underline{D}} = \underline{\underline{g}}\,\underline{\underline{g}}^T$ denoting the corresponding diffusion matrix. If we require in addition the noise to be Gaussian, then the two-point correlations entirely determine the stationary behaviour of the system. In fact, eqs.(1.1) and (1.12) define a stochastic process (see, e.g., [37] for a detailed rigorous account). Here we just concentrate on the formal computation of correlation functions and power spectral densities. The simplest approach is along the lines of signal processing and uses a Fourier representation of the noise

$$\underline{\xi}(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \underline{\xi}_\omega \exp(i\omega t) d\omega\,. \tag{1.13}$$

The expansion coefficients are assumed to be uncorrelated Gaussian random variables $\langle \underline{\xi}_\omega \underline{\bar{\xi}}^T_{\omega'}\rangle = \delta(\omega-\omega')\underline{\underline{I}}$ to ensure Gaussian white noise. The condition $\underline{\xi}_\omega = \underline{\bar{\xi}}_{-\omega}$ keeps the noise real-valued. The formal Fourier transform of eq.(1.1) immediately results in

$$\underline{x}_\omega = \left(i\omega \underline{\underline{I}} - \underline{\underline{A}} - \underline{\underline{B}} \exp(-i\omega\tau)\right)^{-1} \underline{\underline{g}}\,\underline{\xi}_\omega \tag{1.14}$$

where we implicitly assume that all the eigenvalues of the homogeneous equation have negative real part and that we just focus on the stationary properties of the process. Using the properties of the Fourier coefficients $\underline{\xi}_\omega$ the correlation matrix is then easily evaluated as

$$\begin{aligned} \langle \underline{x}(t)\underline{x}^T(t')\rangle &= \frac{1}{2\pi} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \langle \underline{x}_\omega \underline{x}^T_{\omega'} \exp(i\omega t + i\omega' t') d\omega d\omega' \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \underline{\underline{\Sigma}}_\omega \exp(i\omega(t-t'))d\omega \end{aligned} \tag{1.15}$$

where we have introduced the resolvent matrix

$$\underline{\underline{\Sigma}}_\omega = \left(i\omega\underline{\underline{I}} - \underline{\underline{A}} - \underline{\underline{B}}\exp(-i\omega\tau)\right)^{-1} \underline{\underline{D}} \left(-i\omega\underline{\underline{I}} - \underline{\underline{A}}^T - \underline{\underline{B}}^T \exp(i\omega\tau)\right)^{-1}\,. \tag{1.16}$$

The diagonal elements of the matrix (1.16) are just the power spectral densities of the individual components, whereas the off-diagonal elements are the Fourier transform of the corresponding cross correlation function. It is of course obvious that the time delay enters in the same way as in the eigenvalue problem (1.2), and that the eigenvalues determine the complex poles of the resolvent matrix.

1.3 Weakly nonlinear analysis of time-delayed feedback control

Time-delayed feedback control [38, 39] is a convenient setup to control the stability of periodic solutions. As a paradigmatic example we will use a Duffing-Van der Pol oscillator to illustrate various aspects of the analytic perturbation schemes. The equations of motion are given by

$$\begin{aligned} \dot{x}_1(t) &= \omega x_2(t) \\ \dot{x}_2(t) &= -\omega x_1(t) + \varepsilon(\sigma x_1(t) - \nu x_1^3(t) - \mu x_2(t)(1-x_1^2(t))) \\ &\quad -\kappa(x_1(t) - x_1(t-\tau))\,. \end{aligned} \tag{1.17}$$

Here ω denotes the linear frequency, ν governs the nonlinear part of the potential, and μ the nonlinear damping. Time-delayed feedback with control amplitude κ and delay time τ is applied to stabilise periodic orbits with period $T = \tau$, where the equality between period T and time delay τ ensures the control method to be non-invasive. For the purpose of a perturbation expansion ε denotes a small expansion parameter. In the leading order, $\varepsilon = 0$, and without time-delayed feedback, $\kappa = 0$, the system admits a continuum of harmonic solutions, $x_1(t) = |a|\cos(\omega t + \varphi)$, $x_2(t) = -|a|\sin(\omega t + \varphi)$ for any $a \in \mathbb{R}$. Including the nonlinear terms only a few of those, if at all, will result in periodic orbits. Typically the period T of those orbits will depend on nonlinear contributions as well, and multiple scaling techniques or Lindsted type expansions are required to cope with these issues. It is technically simpler, in particular in the presence of time delay, if we include a small detuning σ in the perturbation and require that the resulting orbit, at least at first order, still has period $T = 2\pi/\omega$.

We can perform the perturbation expansion even in a more general setup if we use vector notation. Then eq.(1.17) reads

$$\dot{\underline{x}}(t) = \underline{\underline{A}}\,\underline{x}(t) + \varepsilon \underline{f}(\underline{x}(t)) - \underline{\underline{K}}(\underline{x}(t) - \underline{x}(t-\tau)), \qquad \tau = 2\pi/\omega \tag{1.18}$$

where the linear part admits a pair of imaginary eigenvalues with right- and left-eigenvectors $\underline{u}$ and $\underline{v}^\dagger$, respectively

$$\underline{\underline{A}}\,\underline{u} = i\omega\underline{u}, \quad \underline{v}^\dagger\underline{\underline{A}} = i\omega\underline{v}^\dagger\,. \tag{1.19}$$

1.3.1 Computation of periodic orbits

For $\varepsilon = 0$ eq.(1.18) clearly admits harmonic solutions $a\exp(i\omega t)\underline{u} + c.c.$. Hence, for the nonlinear system we first seek to compute periodic solutions by a simple series expansion

$$\begin{aligned} \underline{x}_*(t) &= \underline{x}^{(0)}(t) + \varepsilon\underline{x}^{(1)}(t) + \ldots \\ \underline{x}^{(0)}(t) &= a\exp(i\omega t)\underline{u} + \bar{a}\exp(-i\omega t)\bar{\underline{u}} \\ \underline{x}^{(1)}(t) &= \underline{x}^{(1)}(t+T)\,. \end{aligned} \tag{1.20}$$

Eq.(1.18) in first order gives the inhomogeneous system

$$\dot{\underline{x}}^{(1)}(t) = \underline{\underline{A}}\,\underline{x}^{(1)}(t) + \underline{f}(\underline{x}^{(0)}(t))\,. \tag{1.21}$$

In the case considered here, $T = \tau$, the delay term drops from this existence condition because the control term is finally non-invasive. Since the solution of eq.(1.21) has to be periodic, see eq.(1.20), a solvability or Fredholm condition applies. The condition can be derived straightforwardly if we multiply eq.(1.21) with $\exp(-i\omega t)\underline{v}^\dagger$, the solution of the adjoint problem. Then the time derivative and the linear contribution combine to result in the total derivative of $\exp(-i\omega t)\underline{v}^\dagger\underline{x}^{(1)}(t)$ and integration over the period $T = \tau = 2\pi/\omega$ yields

$$0 = \int_0^\tau \frac{d}{dt}\left(\exp(-i\omega t)\underline{v}^\dagger\underline{x}^{(1)}(t)\right)dt = \int_0^\tau \exp(-i\omega t)\underline{v}^\dagger\underline{f}(\underline{x}^{(0)}(t))dt\,. \tag{1.22}$$

The condition, eq.(1.22), can be expressed in terms of an effective drift

$$0 = \frac{1}{\tau \underline{v}^\dagger \underline{u}} \int_0^\tau \exp(-i\omega t) \underline{v}^\dagger \underline{f}(\underline{x}^{(0)}(t)) dt = aF(|a|^2)\,. \tag{1.23}$$

A simple phase argument shows that the integral in eq.(1.23) depends only on the modulus $|a|$ apart from a single amplitude factor. If we denote by F_R and F_I the real and the imaginary part of F, respectively, then the condition for the periodic orbit results in

$$F_R(|a|^2) = 0, \quad F_I(|a|^2) = 0\,. \tag{1.24}$$

The first of these conditions determines the value of the amplitude of the periodic orbit, eq.(1.20), whereas the second equation states the constraint that the period of this orbit is not renormalised by the nonlinear contributions. Such a constraint can be, for instance, satisfied by including a small linear part $\varepsilon \underline{\underline{A}}^{(1)} \underline{x}$ in the perturbation, which properly renormalises the linear frequency. In addition, there exists of course the trivial solution $a = 0$ resulting in a small amplitude periodic orbit according to eq.(1.20). For the system without delayed feedback, $\underline{\underline{K}} = 0$, the expression F_R acts as the derivative of an effective potential and the stability of the orbit is directly related to the derivative of F_R.

For our model system, eq.(1.17), the eigenvectors in eq.(1.19) are given by $\underline{u} = (1, i)^T$ and $\underline{v}^\dagger = (1, -i)$. From the definition (1.23) and the perturbation according to eq.(1.17) we obtain

$$F(|a|^2) = F_R(|a|^2) + iF_I(|a|^2) = -\mu(1 - |a|^2)/2 + i(3\nu|a|^2 - \sigma)/2\,. \tag{1.25}$$

Hence the finite amplitude periodic orbit has amplitude $|a| = 1$. It will turn out to be an unstable orbit for $\mu > 0$ as one expects to be the case for a weakly nonlinear Van der Pol oscillator. The second condition in eq.(1.24) on the frequency renormalisation requires $\sigma = 3\nu$. In fact, the nonlinear potential part of the Duffing oscillator introduces a chirp (anisochronicity), a dependence of the period on the amplitude of the oscillation, which is then compensated by the appropriate detuning σ.

1.3.2 Linear stability and strongly stable domain

So far the time delay has not played any essential role for the existence of periodic orbits, because of the noninvasive character of the feedback. Of course, the situation is different when stability considerations become relevant. Assume the periodic orbit, eq.(1.20), is known. Linear stability is governed by the variational equation, which in turn using an exponential ansatz, $\delta \underline{x}(t) = \exp(\Lambda t) \underline{w}(t)$, can be converted into a Floquet eigenvalue problem

$$\dot{\underline{w}}(t) + \Lambda \underline{w}(t) = (\underline{\underline{A}} + \varepsilon D\underline{f}(\underline{x}_*(t)) - \underline{\underline{K}}(1 - \exp(-\Lambda\tau)))\underline{w}(t), \quad \underline{w}(t) = \underline{w}(t+\tau)\,. \tag{1.26}$$

The symbol $D\underline{f}$ denotes the Jacobian matrix. If we consider eq.(1.26) in lowest order, i.e., for $\varepsilon = 0$, then the time dependence drops and the equation reduces to an ordinary (nonlinear) eigenvalue problem. The exponents in this order are determined by the usual quasipolynomial (see e.q.(1.3))

$$\det\left(\underline{\underline{A}} - \Lambda \underline{\underline{I}} - \underline{\underline{K}}(1 - \exp(-\Lambda\tau))\right) = 0\,. \tag{1.27}$$

The condition that all the solutions of eq.(1.27) have nonpositive real part is a necessary constraint for the stability of the orbit. If we use $\Lambda = i\Omega$ in eq.(1.27) we are able to determine the stability boundaries in the parameter space. Our stability condition so far is determined by the control matrix and the dominant linear part of the dynamics, and thus essentially reflects the stability of the control loop. In the leading order, $\varepsilon = 0$, neither the nonlinear part of the dynamics nor the actual shape of the orbit has entered the analysis.

For our example, eq.(1.17), the characteristic equation (1.27) in leading order reads

$$0 = \det \begin{pmatrix} -\Lambda & \omega \\ -\omega - \kappa(1 - \exp(-\Lambda\tau)) & -\Lambda \end{pmatrix} = \Lambda^2 + \omega^2 + \kappa\omega(1 - \exp(-\Lambda\tau)) \,. \quad (1.28)$$

If we choose $\Lambda = i\Omega$ to determine the boundaries of the control domain, it is evident that $\Omega\tau = \pi + 2\pi n$, $n \in \mathbb{Z}$. Hence we obtain from eq.(1.28) the thresholds $\kappa = \omega(n+3/2)(n-1/2)/2$. The stable interval where eq.(1.28) has no solution with positive real part is given by

$$\kappa \in (-3\omega/8, 5\omega/8) \,. \quad (1.29)$$

Outside this interval there exists at least one Floquet exponent with positive real part of order one. As our stability condition has been derived in leading order we call eq.(1.29) the strongly stable domain.

1.3.3 Perturbation expansion of the eigenvalue problem and weak instabilities

Floquet exponents which in leading order already have a nonvanishing real part do not change the stability properties if the small perturbation is taken into account. Hence the strongly stable domain, eq.(1.29), is a necessary constraint for stability. However, within this interval and in order $\varepsilon = 0$ there still occur two neutral modes with leading Floquet exponent zero. It is easy to verify that

$$\underline{w}^{(0)}(t) = \alpha \exp(i\omega t)\underline{u} + \bar{\alpha} \exp(-i\omega t)\bar{\underline{u}} \quad (1.30)$$

solves eq.(1.26) for $\varepsilon = 0$ and $\Lambda = 0$. Eq.(1.30) determines a two-dimensional subspace, parametrised by α and $\bar{\alpha}$, which contains the Goldstone mode of the periodic orbit. That means no matter what kind of perturbation we apply one of the Floquet exponents remains zero. However, the second exponent may become nonzero and thus results in an additional stability condition. We aim for computing such a Floquet exponent using the straightforward series expansions

$$\underline{w}(t) = \underline{w}^{(0)}(t) + \varepsilon \underline{w}^{(1)}(t) + \ldots, \qquad \Lambda = 0 + \varepsilon\Lambda^{(1)} + \ldots \,. \quad (1.31)$$

Then eq.(1.26) in first order reads

$$\dot{\underline{w}}^{(1)}(t) = \underline{\underline{A}}\, \underline{w}^{(1)}(t) + \underline{h}(t), \quad \underline{w}^{(1)}(t) = \underline{w}^{(1)}(t+\tau) \quad (1.32)$$

with inhomogeneous part

$$\underline{h}(t) = \left(D\underline{f}(\underline{x}^{(0)}(t)) - \Lambda^{(1)} - \Lambda^{(1)}\tau\underline{\underline{K}}\right) \underline{w}^{(0)}(t) \,. \quad (1.33)$$

As in the previous section the existence of a periodic solution of eq.(1.32) puts secular constraints on the inhomogeneous part, which can be easily derived if we use the neutral modes of the adjoint problem, $\exp(-i\omega t)\underline{v}^\dagger$ and $\exp(i\omega t)\underline{\bar{v}}^\dagger$. If we multiply eq.(1.32) with one of these, the time derivative and the linear term combine to give a total derivative and integration over one period finally yields the secular conditions

$$\int_0^\tau \exp(-i\omega t)\underline{v}^\dagger \underline{h}(t)dt = 0, \quad \int_0^\tau \exp(i\omega t)\underline{\bar{v}}^\dagger \underline{h}(t)dt = 0\,. \tag{1.34}$$

In view of eq.(1.30) and (1.33) these conditions constitute a two-dimensional linear system for α and $\bar{\alpha}$. The vanishing of the determinant yields the condition for the eigenvalue $\Lambda^{(1)}$ at first order. The integrals occurring in eq.(1.34) can, in fact, be written in terms of the previously introduced effective drift, eq.(1.23). If we take in eq.(1.23) derivatives with respect to a or $\bar{a}$ we obtain

$$\begin{aligned} a^2F'(|a|^2) &= \int_0^\tau \exp(-i\omega t)\underline{v}^\dagger D\underline{f}(\underline{x}^{(0)}(t))\underline{\bar{u}}\exp(-i\omega t)dt/(\tau\underline{v}^\dagger\underline{u}) \\ F(|a|^2)+|a|^2F'(|a|^2) &= \int_0^\tau \exp(-i\omega t)\underline{v}^\dagger D\underline{f}(\underline{x}^{(0)}(t))\underline{u}\exp(i\omega t)dt/(\tau\underline{v}^\dagger\underline{u})\,. \end{aligned} \tag{1.35}$$

Employing the property (1.24) the two secular conditions, eq.(1.34), can be written as the homogeneous system

$$\begin{pmatrix} |a|^2F'(|a|^2)-\Lambda^{(1)}-\gamma\Lambda^{(1)} & a^2F'(|a|^2) \\ \bar{a}^2\bar{F}'(|a|^2) & |a|^2\bar{F}'(|a|^2)-\Lambda^{(1)}-\bar{\gamma}\Lambda^{(1)} \end{pmatrix}\begin{pmatrix}\alpha\\ \bar{\alpha}\end{pmatrix} = 0\,, \tag{1.36}$$

where the abbreviation

$$\gamma = \tau\underline{v}^\dagger\underline{\underline{K}}\,\underline{u}/(\underline{v}^\dagger\underline{u}) \tag{1.37}$$

takes the effect of the control loop into account. Eq.(1.36) finally results in the characteristic polynomial

$$0 = \left(\Lambda^{(1)}\right)^2|1+\gamma|^2 - 2\Lambda^{(1)}\mathrm{Re}\left((1+\bar{\gamma})|a|^2F'(|a|^2)\right) \tag{1.38}$$

which determines the two small Floquet exponents in first order. Clearly one of the solutions $\Lambda^{(1)}=0$ corresponds to the Goldstone mode, while the other may take nontrivial values and may induce a weak instability of the orbit. Without control $\gamma=0$ we have $\Lambda^{(1)}=2|a|^2F_R'(|a|^2)$ which, as already mentioned above, determines the stability properties of the periodic orbit without delayed feedback. In cases when the coefficient of the leading term in eq.(1.38) vanishes the results have to be considered with some care.

For our model, eq.(1.17), the effective drift, eq.(1.25), gives $F'(|a|^2)=(\mu+3i\nu)/2$ so that the orbit with amplitude $|a|=1$ and without delayed feedback $\kappa=0$ is unstable for $\mu>0$, as already mentioned. With the eigenvectors stated above we obtain from eq.(1.37) $\gamma=-i\tau\kappa/2$, and the polynomial (1.38) results in the nontrivial eigenvalue $\Lambda^{(1)}|1+i\tau\kappa/2|^2=\mu-3\tau\kappa\nu/2$. Thus the weak stability condition induced by this branch reads

$$\kappa\nu > \mu\omega/(3\pi)\,. \tag{1.39}$$

In this particular example a nonzero value for the nonlinear potential, i.e., a chirp is required for stabilisation. The two conditions, eq.(1.29) and (1.39), give a fairly simple shape for the control domain in the parameter space. The perturbative results are in quite good agreement with numerical results obtained for finite but small

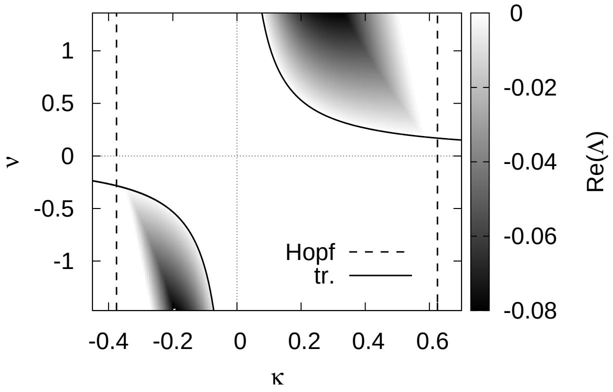

Fig. 1.1. Control domain for the periodic orbit of the Duffing-Van der Pol oscillator, eq.(1.17), for $\mu = 1.0$, $\omega = 1.0$, $\sigma = 3\nu$, and delay τ coinciding with the period of the orbit. Dashed lines: analytic result of the strong stability boundary, eq.(1.29). Solid lines: analytic result for the weak stability boundary, eq.(1.39). Shading indicates numerical results for the real part of the leading Floquet exponent for $\varepsilon = 0.05$.

values of the expansion parameter, see figure 1.1. Overall the perturbative results can provide some insight which type of feedback could be used to achieve stabilisation, even though the analytic expressions are limited to the weakly nonlinear regime

The weakly nonlinear analysis of the model (1.17) has very much in common with the discussion of the corresponding Hopf normal form (Stuart Landau oscillator) subjected to time delayed feedback [40–42]. The analysis provided here is able to link the control domain with the actual parameters of the original equation of motion. In fact, the procedure applied to our example can be almost verbatim transferred to the discussion of the general weakly nonlinear system

$$\dot{\underline{x}}(t) = \underline{\underline{A}}\,\underline{x}(t) + \underline{\underline{B}}\,\underline{x}(t-\tau) + \varepsilon \underline{f}(\underline{x}(t), \underline{x}(t-\tau)) \qquad (1.40)$$

if we assume that the linear part, i.e., the equation for $\varepsilon = 0$ supports periodic solutions.

1.4 Coherence resonance modulated by time-delayed feedback

The counterintiuitive phenomenon of coherence resonance [4, 43] was originally discovered for excitable systems. It implies that noise-induced oscillations become most regular for an optimum non-zero value of the noise intensity. It has been shown that coherence resonance can be enhanced or suppressed by applying time-delayed feedback in systems with type-I [44] and type-II [45] excitability. Recently, coherence resonance has also been found in non-excitable systems with a subcritical Hopf bifurcation [46–48] and its modulation by time-delayed feedback has been demonstrated theoretically for a subcritical Hopf normal form [29], and confirmed in experiments with an electronic circuit for a generalised Van der Pol system [49]. It is important to note that the pure coherence resonance effect for non-excitable systems is observed for a subcritical Hopf bifurcation and not for the supercritical case. The standard Van der Pol model close to a supercritical Hopf bifurcation has been investigated in the presence of delay and noise [45,50,51], but here no coherence resonance, i.e., no non-monotonic dependence of the correlation time upon the noise intensity, is found. Coherence resonance has also been observed in real-world systems, for instance, in microwave dynamical systems such as a five-cavity delayed-feedback klystron oscillator at the self-excitation threshold [52], in lasers with saturable absorber [53], with optical feedback [54–56], with optical injection [57], or in semiconductor superlattices [58,59].

Here we investigate the impact of time-delayed feedback in a non-excitable model of coherence resonance. We will apply perturbation techniques to a model which can be considered as the normal form of a subcritical Hopf bifurcation, see [29]. The calculation of the stationary amplitude probability distribution and of correlation functions will be at the centre of interest. Consider a two-variable quintic normal form of a nonlinear oscillator

$$\begin{aligned}
\dot{x}_1(t) &= (\lambda + r^2(t) - r^4(t))x_1(t) - 2\pi x_2(t) - K(x_1(t) - x_1(t-\tau)) + \sqrt{2D}\xi_1(t) \\
\dot{x}_2(t) &= (\lambda + r^2(t) - r^4(t))x_2(t) + 2\pi x_1(t) - K(x_2(t) - x_2(t-\tau)) + \sqrt{2D}\xi_2(t) \\
r^2(t) &= x_1^2(t) + x_2^2(t)
\end{aligned} \tag{1.41}$$

subjected to time-delayed feedback with control amplitude K and to isotropic Gaussian white noise of strength D, with vanishing mean $\langle \xi_k(t) \rangle = 0$ and correlation function $\langle \xi_k(t)\xi_\ell(t') \rangle = \delta(t-t')\delta_{k\ell}$. The timescale has been adjusted such that the linear frequency is given by $\omega = 2\pi$, and λ denotes the bifurcation parameter. For simplicity of the perturbative treatment we will only allow integer values of the delay τ. The system mimics a subcritical Hopf bifurcation, including the saddle node bifurcation of the unstable limit cycle at $\lambda = -1/4$, which is the main cause for the coherence resonance phenomenon.

1.4.1 Adiabatic elimination and stationary probability distribution

To derive an effective simple stochastic differential equation let us first analyse the bifurcations of the trivial stationary state $x_1 = x_2 = 0$ of the deterministic part of the model (1.41) . The variational equation results in an eigenvalue problem of the form eq.(1.2) and the characteristic equation (1.3) reads

$$\begin{aligned}
0 &= \det \begin{pmatrix} \lambda - \Lambda - K(1-\exp(-\Lambda\tau)) & -2\pi \\ 2\pi & \lambda - \Lambda - K(1-\exp(-\Lambda\tau)) \end{pmatrix} \\
&= (\lambda - \Lambda - 2\pi i - K(1-\exp(-\Lambda\tau)))\,(\lambda - \Lambda + 2\pi i - K(1-\exp(-\Lambda\tau)))
\end{aligned} \tag{1.42}$$

Obviously, for integer values for the delay τ and $\lambda = 0$, $\Lambda = \pm i\Omega = \pm 2\pi i$ is a purely imaginary pair of eigenvalues, giving rise to a Hopf bifurcation at $\lambda = 0$. It is in fact possible to show that all the other eigenvalues have negative real part if K is sufficiently small, for instance, the condition $|K\tau| \leq 1$ is sufficient. But the corresponding techniques are fairly nontrivial and a discussion of the related algebraic and numerical concepts can be found, e.g., in [10, 19, 60]. We can apply the techniques to derive the effective stochastic differential equation, eq.(1.11), when rewriting the model (1.41) using the notation of eq.(1.8)

$$\underline{\underline{A}} = \begin{pmatrix} -K & -2\pi \\ 2\pi & -K \end{pmatrix} \quad , \quad \underline{\underline{B}} = \begin{pmatrix} K & 0 \\ 0 & K \end{pmatrix}$$
$$\underline{f}(\underline{x}(t), \underline{x}(t-\tau)) = (\lambda + r^2(t) - r^4(t)) \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$$
$$\underline{\underline{g}}(\underline{x}(t), \underline{x}(t-\tau)) = \sqrt{2D} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} . \tag{1.43}$$

For $\Lambda = i\Omega = 2\pi i$ the eigenvectors of the linear part (see eqs.(1.2) and (1.4)) are easily computed as $\underline{u}_{i\Omega} = (1, -i)^T$ and $\underline{v}^\dagger_{i\Omega} = (1, i)$. Eq.(1.9) then tells us that on the slow manifold the phase space variables can be expressed in terms of the amplitude $C(t)$

$$\begin{aligned} x_1(t+\theta) &= C(t)\exp(i2\pi\theta) + \bar{C}(t)\exp(-i2\pi\theta) + R_1(C(t), \theta) \\ x_2(t+\theta) &= -iC(t)\exp(i2\pi\theta) + i\bar{C}(t)\exp(-i2\pi\theta) + R_2(C(t), \theta) . \end{aligned} \tag{1.44}$$

In particular, neglecting the nonlinear contributions in eq.(1.44) we have $x_1(t) = 2\mathrm{Re}(C(t)) + \ldots$, $x_2(t) = 2\mathrm{Im}(C(t)) + \ldots$ and $r^2(t) = 4|C(t)|^2 + \ldots$. Thus, the amplitude C has a a direct meaning in terms of the original phase space values. Finally, using the higher order terms given in eq.(1.43) the effective stochastic differential equation (1.11) reads

$$\dot{C}(t) = 2\pi i C(t) + \frac{(\lambda + |2C(t)|^2 - |2C(t)|^4)2C(t) + \sqrt{2D}(\xi_1(t) + i\xi_2(t))}{2(1+K\tau)} \tag{1.45}$$

where we have evaluated the nonlinear terms using the lowest order linear approximation on the slow manifold, eq.(1.44). The time-delayed feedback results in an effective rescaling of the time scale, which in turn can be converted to a rescaling of the noise intensity D. As the system (1.45) has a phase symmetry, the stationary distribution is spherically symmetric as well and can be computed by standard methods quite easily [8]. If we use the radial variable $r = |C|/2$ the stationary distribution reads

$$P(r) = Nr \exp\left(\frac{r^2(\lambda/2 + r^2/4 - r^6/6)}{D/(1+K\tau)} \right) \tag{1.46}$$

where N denotes the normalisation factor. The analytic approximation, which has been obtained as an expansion at the bifurcation point, gives in fact a rather accurate description even if parameter values deviate substantially from the bifurcation point (see figure 1.2).

The formal derivation of the effective stochastic model (1.45) has not been entirely systematic, as we were including higher order nonlinear terms to keep the system globally stable. Such kind of heuristic approach is often used in the physics literature, and can in principle be dealt with by computing the higher order nonlinear corrections to the slow manifold (1.44).

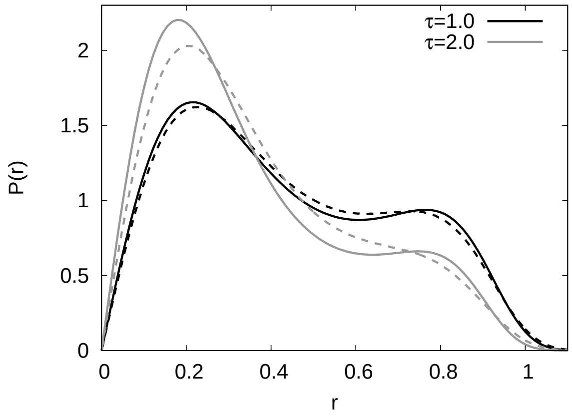

Fig. 1.2. Stationary probability distribution of the radial variable $r = (x_1^2 + x_2^2)^{1/2}$ for the stochastic delay system, eq.(1.41), with parameters $\lambda = -0.26$, $K = 0.5$, $D = 0.015$, and different values of the delay τ. Solid lines: distribution according to the analytic perturbation expansion, eq.(1.46). Dashed lines: results obtained from numerical simulations.

1.4.2 Dynamical correlations and power spectral density

Correlation functions and power spectral densities are at the heart of detecting coherence resonance phenomena. The evaluation of dynamical correlation functions in nonlinear stochastic systems is an inherently difficult task. If time delay is added then one cannot even resort to spectral theories, e.g., using Fokker-Planck operators. On the contrary, the evaluation of correlations in linear systems is almost trivial by comparison. Hence, one often aims at approximating the original nonlinear dynamics by a suitable linear system [61], even at the expense of a systematic approximation scheme. Following ideas developed in the context of Fokker-Planck systems with nonlinear drift [62] one tries to approximate nonlinear terms in the equations of motion by linear ones, using some kind of optimisation approach. To illustrate the main idea let us consider the stochastic time delay equation

$$\dot{\underline{x}}(t) = \underline{f}(\underline{x}(t)) + \underline{\underline{B}}\,\underline{x}(t-\tau) + \underline{\underline{g}}\,\underline{\xi}(t) \tag{1.47}$$

which covers as well the model of interest, eq.(1.41) when we choose the nonlinear term $\underline{f}(\underline{x})$ appropriately. We intend to replace this nonlinear function by a linear contribution $\underline{\underline{A}}\,\underline{x}$ by making the "best choice" for the coefficient matrix in the sense that we minimise the mean square deviation $\langle(\underline{f}(\underline{x}) - \underline{\underline{A}}\,\underline{x})^T(\underline{f}(\underline{x}) - \underline{\underline{A}}\,\underline{x})\rangle$. The minimisation yields

$$\underline{\underline{A}} = \langle \underline{f}(\underline{x})\underline{x}^T\rangle\langle\underline{x}\,\underline{x}^T\rangle^{-1} \tag{1.48}$$

with eq.(1.1) and (1.12) being the "best" linear approximation of eq.(1.47). The evaluation of eq.(1.48) requires the computation of static expectation values, a feature which is quite common in dynamical theories of this kind. One could deduce such values either from simulations or from alternative theories, like, e.g., those developed in the previous section. The impact of the nonlinearity on the dynamics has been effectively condensed in the few parameters, eq.(1.48). The computation of dynamical correlation functions now becomes a trivial task as we can resort to the exact result available for linear systems, see eqs.(1.15) and (1.16). There are certainly alternatives to arrive at a suitable linear model, i.e., the scheme outlined here is by no means a unique way to solve the task. In addition, the accuracy of the method is difficult to predict a priori. Like most mean-field schemes the approach does not rely on a small expansion parameter. But the scheme has the potential to capture at least qualitatively the main features of the correlations as the impact of the time delay has been fully taken into account.

For our model (1.41) we have, using the notation of eq.(1.47)

$$\underline{\underline{B}} = \begin{pmatrix} K & 0 \\ 0 & K \end{pmatrix}, \quad \underline{\underline{g}} = \sqrt{2D} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{1.49}$$

while the components of the nonlinear part are given by $f_1 = (\lambda + r^2 - r^4 - K)x_1 - 2\pi x_2$ and $f_2 = (\lambda + r^2 - r^4 - K)x_2 + 2\pi x_1$. Since our model is spherically symmetric all static cross correlations vanish, i.e., $\langle x_k x_\ell \rangle = 0$ and $\langle (r^2 - r^4) x_k x_\ell \rangle = 0$ for $k \neq \ell$. The diagonal components coincide and expectation values can be written in terms of the radial variable, $\langle x_k x_k \rangle = \langle r^2 \rangle / 2$ and $\langle (r^2 - r^4) x_k x_k \rangle = \langle (r^2 - r^4) r^2 \rangle / 2$. Thus, eq.(1.48) results in

$$\underline{\underline{A}} = \begin{pmatrix} \lambda_{eff} - K & -2\pi \\ 2\pi & \lambda_{eff} - K \end{pmatrix}, \quad \lambda_{eff} = \lambda + \langle (r^4 - r^6) \rangle / \langle r^2 \rangle \,. \tag{1.50}$$

Within our approximation the entire impact of the nonlinear contributions is a static renormalisation of the bifurcation parameter. The required expectation values can be either obtained from the approximation, eq.(1.46), or from simulations. One can also aim at a self-consistent computation using the Gaussian stationary solution of the effective model, eqs.(1.1) and (1.12), even though the actual stationary density is by no means well approximated by a normal distribution, see figure 1.2. Finally, the correlation matrix is easily evaluated from eq.(1.16). The result is conveniently stated if we introduce the abbreviation

$$z(\omega) = i\omega - \lambda_{eff} + K(1 - \exp(-i\omega\tau)) \,. \tag{1.51}$$

The diagonal elements coincide, $(\Sigma_\omega)_{11} = (\Sigma_\omega)_{22} = S(\omega)$, and represent the power spectral density of each component

$$S(\omega) = 2D \frac{|z(\omega)|^2 + (2\pi)^2}{|z(\omega) - 2\pi i|^2 |z(\omega) + 2\pi i|^2} = D(|z(\omega) - 2\pi i|^{-2} + |z(\omega) + 2\pi i|^{-2}) \,. \tag{1.52}$$

For the cross correlations we obtain

$$\begin{aligned} (\Sigma_\omega)_{12} = -(\Sigma_\omega)_{21} &= 2D \frac{2\pi(z(\omega) - z(-\omega))}{|z(\omega) - 2\pi i|^2 |z(\omega) + 2\pi i|^2} \\ &= 2D \frac{4\pi i(\omega + K \sin(\omega\tau))}{|z(\omega) - 2\pi i|^2 |z(\omega) + 2\pi i|^2} \,. \end{aligned} \tag{1.53}$$

The results of this phenomenological approximation scheme are quite encouraging as the main features of the power spectral density are captured quantitatively to a fairly high degree of precision, see figure 1.3.

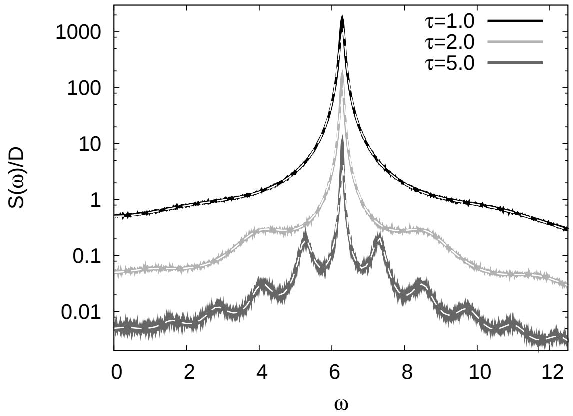

Fig. 1.3. Normalised power spectral density $S(\omega)/D$ of the model, eq.(1.41), for $\lambda = -0.26$, $K = 0.5$, $D = 0.015$ and different values of the delay. Solid lines: numerical simulations of the stochastic delay differential equation. Dashed lines (white): analytic approximation according to eq.(1.52). For better visibility the top and the bottom spectrum is shifted by ± 10dB, respectively.

1.5 Concluding remarks

Formal analytic expansion schemes, even if they just cover very simple setups, play their role for the investigation of dynamical systems, in particular for models including noise and time delay. Initially, the methods have been largely developed within an engineering context for ordinary differential equations. The tools turn out to be useful for the understanding of basic mechanisms in dynamical systems and provide an analytic overview of structures in phase- and parameter space. Even for contemporary research questions, such as the relevance of coupling topologies in networks with noise or time delay, or global dynamical aspects such as manifolds determining basins of attraction, simple analytic perturbation schemes could give viable input. For instance, Melnikov's method has been empirically generalised to cope with noise and time delay, but systematic studies are still lacking.

Our selection of perturbation expansion schemes was of course incomplete, and there are plenty of related approaches available in the literature. Essentially all of these have limitations from a mathematical point of view. Unlike the impression which is sometimes given in the physics literature formal expansions do not provide rigorous proofs. Caveats apply as well when applications are considered. Most real world problems do not come in a setting which allows for the application of analytic expansion schemes. Hence, numerical approaches, say either direct simulations or sophisticated continuation techniques to track bifurcations, are certainly often the method of choice to determine the behaviour of a particular dynamical systems. Having said that, simple analytic formal expansion schemes such as those sketched above, combined with rigorous methods or numerical approaches can provide useful insight for complex dynamical behaviour of systems with noise and time delay.

Acknowledgement

This work was supported by DFG in the framework of SFB 910.

References

1. A. Einstein, Ann. Phys. **322**, 549 (1905)
2. H.A. Kramers, Physica **7**, 284 (1940)
3. P. Hänggi, P. Talkner, M. Borkovec, Rev. Mod. Phys. **62**, 251 (1990)
4. A. Pikovsky, J. Kurths, Phys. Rev. Lett. **78**, 775 (1997)
5. P. Gammaitoni, P. Hänggi, P. Jung, F. Marchesoni, Rev. Mod. Phys. **70**(1), 223 (1998)
6. R. Klages, W. Just, C.Jarzynski (eds.), *Nonequilibrium Statistical Physics of Small Systems: Fluctuation Relations and Beyond* (Wiley-VCH, Weinheim, 2013)
7. W. Horsthemke, R. Lefever, *Noise-Induced Transitions* (Springer, Berlin, 1984)
8. H. Risken, *The Fokker-Planck equation : methods of solution and applications* (Springer, Berlin, 1989)
9. L. Arnold, *Random Dynamical Systems* (Springer, Berlin, 2002)
10. R. Bellmann, K.L. Cooke, *Differential–Difference Equations* (Acad. Press, New York, 1963)
11. E. Fick, G. Sauermann, *The quantum statistics of dynamic processes* (Springer, Berlin, 1990)
12. R. Kubo, M. Toda, N. Hashitsume, *Statistical physics 2. Nonequilibrium statistical mechanics* (Springer, Berlin, 1991)
13. T. Erneux, *Applied delay differential equations* (Springer, Berlin, 2009)
14. F.M. Atay (ed.), *Complex Time-Delay Systems* (Springer, Berlin, Heidelberg, 2010)
15. W. Just, A. Pelster, M. Schanz, E. Schöll (eds.), *Delayed Complex Systems*, Theme Issue of Phil. Trans. R. Soc. A **368**, 301-513 (2010)
16. V. Flunkert, I. Fischer, E. Schöll (eds.), *Dynamics, control and information in delay-coupled systems*, Theme Issue of Phil. Trans. R. Soc. A **371**, 20120465 (2013)
17. J.Q. Sun, G. Ding (eds.), *Advances in Analysis and Control of Time-Delayed Dynamical Systems* (World Scientific, Singapore, 2013)
18. M. Soriano, J. Garca-Ojalvo, C. Mirasso, I. Fischer, Rev. Mod. Phys. **85**, 421 (2013)
19. J.K. Hale, S.M. Verduyn Lunel, *Introduction to Functional Differential Equations* (Springer, New York, 1993)
20. K. Engelborghs, *DDE-BIFTOOL: a Mathlab package for bifurcation analysis of delay differential equations.* *http://www.cs.kuleuven.ac.be/koen/delay/ddebiftool.shtml*
21. L. Pecora, T. Carroll, Phys. Rev. Lett. **80**, 2109 (1998)
22. S. Yanchuk, M. Wolfrum, P. Hövel, E. Schöll, Phys. Rev. E **74**, 026201 (2006)
23. M. Wolfrum, S. Yanchuk, P. Hövel, E. Schöll, Eur. Phys. J. Special Topics **191**, 91 (2010)
24. T.D. Frank, P.J. Beck, R. Friedrich, Phys. Rev. E **68**, 021912 (2003)
25. T. Frank, Phys. Rev. E **72**, 011112 (2005)

26. A. Nayfeh, *Perturbation Methods* (Wiley-VCH, Weinheim, 2000)
27. M.C. Cross, P.C. Hohenberg, Rev. Mod. Phys. **65**(3), 851 (1993)
28. W. Just, H. Benner, C. v. Loewenich, Physica D **199**, 33 (2004)
29. P. M. Geffert, A. Zakharova, A. Vüllings, W.Just, E. Schöll, Eur. Phys. J. B **87**, 291 (2014)
30. H. Haken, *Synergetics: Introduction and Advanced Topics* (Springer, Berlin, 2004)
31. A. Halanay, *Differential equations: stability, oscillations, time lags* (Acad. Press, New York, 1966)
32. A. Amann, E. Schöll, W. Just, Physica A **373**, 191 (2007)
33. R.M. Corless, G.H. Gonnet, D.E.G. Hare, D.J. Jeffrey, D.E. Knuth, Adv. Comp. Math. **5**, 329 (1996)
34. F. Giannakopulos, A. Zapp, Physica D **159**, 215 (2001)
35. B.F. Redmond, V.G. LeBlanc, A. Longtin, Physica D **166**, 131 (2002)
36. N.G. van Kampen, Phys. Rep. **124**(2), 69 (1985)
37. U. Küchler, B. Mensch, Stoch. & Stoch. Rep. **40**, 23 (1992)
38. K. Pyragas, Phys. Lett. A **170**, 421 (1992)
39. E. Schöll, H.G. Schuster (eds.) *Handbook of Chaos Control* (Wiley-VCH, Weinheim, 2008)
40. B. Fiedler, V. Flunkert, M. Georgi, P. Hövel, E. Schöll, Phys. Rev. Lett. **98**, 114101 (2007)
41. W. Just, B. Fiedler, M. Georgi, V. Flunkert, P. Hövel, E. Schöll, Phys. Rev. E **76**, 026210 (2007)
42. G. Brown, C. M. Postlethwaite, M. Silber, Physica D **240**, 859 (2011)
43. G. Hu, T. Ditzinger, C.Z. Ning, H. Haken, Phys. Rev. Lett. **71**, 807 (1993)
44. R. Aust, P. Hövel, J. Hizanidis, E. Schöll, Eur. Phys. J. Spec. Top. **187**, 77 (2010)
45. N.B. Janson, A.G. Balanov, E. Schöll, Phys. Rev. Lett. **93**, 010601 (2004)
46. O.V. Ushakov, H.J. Wünsche, F. Henneberger, I.A. Khovanov, L. Schimansky-Geier, M.A. Zaks, Phys. Rev. Lett. **95**, 123903 (2005)
47. A. Zakharova, T. Vadivasova, V. Anishchenko, A. Koseska, J. Kurths, Phys. Rev. E **81**, 011106 (2010)
48. A. Zakharova, A. Feoktistov, T. Vadivasova, E. Schöll, Eur. Phys. J. Spec. Top. **222**, 2481 (2013)
49. V. Semenov, A. Feoktistov, T. Vadivasova, E. Schöll, A. Zakharova, Chaos **25**, 033111 (2015)
50. E. Schöll, A.G. Balanov, N.B. Janson, A.B. Neiman, Stoch. Dyn. **5**, 281 (2005)
51. J. Pomplun, A. Amann, E. Schöll, Europhys. Lett. **71**, 366 (2005)
52. B. Dmitriev, Y. Zharkov, S. Sadovnikov, V. Skorokhodov, A. Stepanov, Tech. Phys. Lett. **37**, 1082 (2011)
53. J.L.A. Dubbeldam, B. Krauskopf, D. Lenstra, Phys. Rev. E **60**, 6580 (1999)
54. G. Giacomelli, M. Giudici, S. Balle, J.R. Tredicce, Phys. Rev. Lett. **84**, 3298 (2000)
55. J.F.M. Avila, H.L.D. de S. Cavalcante, J.R.R. Leite, Phys. Rev. Lett. **93**, 144101 (2004)
56. C. Otto, B. Lingnau, E. Schöll, K. Lüdge, Opt. Express **22**, 13288 (2014)
57. D. Ziemann, R. Aust, B. Lingnau, E. Schöll, K. Lüdge, Europhys. Lett. **103**, 14002 (2013)
58. J. Hizanidis, E. Schöll, Phys. Rev. E **78**, 066205 (2008)

59. Y. Huang, H. Qin, W. Li, S. Lu, J. Dong, H.T. Grahn, Y. Zhang, Europhys. Lett. **105**, 47005 (2014)
60. M.E. Bleich, J.E.S. Socolar, Phys. Lett. A **210**, 87 (1996)
61. A. Vüllings, E. Schöll, B. Lindner, Eur. Phys. J. B **87**, 31 (2014)
62. J. Kottalam, K. Lindenberg, B.J. West, J. Stat. Phys. **42**, 979 (1986)

Index