Principles of time delayed feedback control

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(January 27, 1998)

The huge number of unstable periodic orbits make chaotic dynamical systems a promising target for control techniques. Delayed feedback methods build up a control force from a simple difference of an output signal at different times. Hence no fancy data processing or modelling of the internal dynamics is required and the application in experimental situations is straightforward. The theoretical foundation of such control schemes is reviewed. No special assumption is imposed on the underlying system or on the coupling of the control force to the degrees of freedom. The analysis clearly shows that only orbits with a finite torsion are accessible for control. Even quantitative results are obtained on a general basis, i. e. the dependence on the control parameters. The limits of simple delayed feedback methods are explained and advanced strategies, which overcome these difficulties are analysed. Finally the problem of the proper adjustment of delay times is addressed too.

I. INTRODUCTION

Control theory is one of the central subjects in engineering science. It has been studied for at least half a century and has profited from a huge formal apparatus developed in applied mathematics (cf. [1]). In fact, the scope of general control theory is very large and covers almost all perturbations of systems, which intend to direct the motion into a desired state.

Within this background one might wonder why physicists have invaded the field of control in the last decade, with an emphasis on nonlinear and especially chaotic dynamical systems. Such an interest came from the neither new observation that chaotic dynamical systems admit a huge number of unstable periodic orbits. Since these orbits represent genuine motions of the system under consideration one can expect to achieve stabilisation by applying tiny control forces. Hence chaotic dynamics opens the possibility to control quite distinct types of motion in a single system with small control power [2]. From this point of view the actual control strategy is of minor importance, and one usually resorts to control methods which are quite elementary compared to the elaborated machinery of conventional control theory mentioned above. Nevertheless, the implications of this idea, which has now become popular under the slightly misleading and counterintuitive notion of chaos control, is tremendous.

Whenever a good theoretical modelling of the dynamics is possible, e. g. like in pure mechanical or electronical systems, then quite sophisticated dynamical features can be controlled by appropriate algorithms and the present book contains a lot of these aspects (cf. also [3]). Here I want to concentrate on the somewhat opposite situation, i. e. that either no successful theoretical modelling is possible or that sophisticated on line data processing fails for reasons of time scales. For both cases experimental paradigms are available. On the one hand one may consider the nonlinear dynamics in high power ferromagnetic resonance experiments [4], which is even to date poorly understood from the theoretical point of view [5]. On the other hand a certain class of semiconductor lasers with optical feedback have been considered recently, where the dynamics takes place on the nanoseconds time scale (cf. [6]). Whereas in the former example control may contribute to the understanding of the internal nonlinear dynamics, it is interesting from the technological point of view in the latter case.

The control methods discussed here should allow to stabilise the potentially large number of periodic orbits in chaotic systems but without modifying the shape of these genuine orbits. For the reasons mentioned in the preceeding paragraph the schemes should be solely based on simple measurements of output data without intensive data processing or modelling of the system. A method which meets these constraints and which will be discussed extensively in the following sections, uses a feedback consisting of the difference between the present output signal and the delayed signal. [7]. Such a general idea is in fact not quite new. In particular the influence of delays has been investigated for a long time in the context of stabilising time independent states. Furthermore models of this kind have been

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discussed in the context of balancing by humans half a century ago [8]. The new aspect, which apparently has not been investigated yet in detail, is related to the control of actual unstable periodic orbits by delay methods.

Successful experimental realisations of control of periodic motions using delayed feedback methods have been reported in the literature. Most of them deal with electronic circuits, which however may be viewed as a kind of toy experiment. There are a few different applications available, namely the control of a CO_2 laser system [9], of the periodic states in a discharge tube [10], and the control of a mechanical oscillating metal beam [11].

The emphasis of the present article is on the general theoretical background of delayed feedback control methods. The development of an analytical understanding of the control method for periodic orbits is here at the centre of interest in contrast to complete discussions of special model systems. In particular, I will present analytical approaches that work irrespective of the special form of the system under consideration but allow for explicit quantitative statements which are in general valid for delayed feedback control schemes. Topics that are dealt with cover the question what kind of orbits is accessible for control, by which bifurcations control is achieved, what mechanism may limit the success of delayed feedback methods, and how the delay time can be adjusted if the period of the orbit is not known a priori. Results of numerical simulations and experimental data, which coincide with the theoretical predictions will be referenced at the appropriate places, but I apologise in advance that such references are far from being complete.

II. MECHANISM OF DELAYED FEEDBACK CONTROL

To keep the approach as general as possible, let me consider a kind of black box experiment, where in particular no information about the equation of motion is required [12]. I just presuppose that the state of the system can be described in principle by a state vector $\boldsymbol{x}(t)$ and that a scalar quantity, $g(\boldsymbol{x}(t))$, which depends on the state of the system is accessible to measurements. The dynamics should admit an unstable periodic orbit $\boldsymbol{\xi}(t) = \boldsymbol{\xi}(t+T)$ with period T, which we intend to stabilise by employing a suitably chosen control force $F(t)$. The equation of motion which fits these constraints reads

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{f}(\boldsymbol{x}(t), K\,F(t)) \quad , \tag{1}$$

where the functional dependence on the control force is not specified yet. The amplitude of the control force K, which of course can be incorporated in the definition of $F(t)$, has been noted separately for later reference. The system under consideration may depend explicitly on time in a periodic fashion, $\boldsymbol{f}_t = \boldsymbol{f}_{t+T}$, but for simplicity in the notation the index is dropped.

Before we are going to dwell on control schemes let me first consider the free dynamics $K = 0$. First of all the existence of the unstable periodic orbit, $\boldsymbol{x}(t) = \boldsymbol{\xi}(t)$, poses the trivial constraint

$$\dot{\boldsymbol{\xi}}(t) = \boldsymbol{f}(\boldsymbol{\xi}(t), 0) \tag{2}$$

on the dynamics. Furthermore, if one performs a formal linear stability analysis according to $\boldsymbol{x}(t) = \boldsymbol{\xi}(t) + \delta\boldsymbol{x}(t)$, one obtains

$$\delta\dot{\boldsymbol{x}}(t) = D_1\boldsymbol{f}(\boldsymbol{\xi}(t), 0)\delta\boldsymbol{x}(t) \quad . \tag{3}$$

Here $D_1\boldsymbol{f}$ denotes the vector derivative with respect to the first argument, i. e. the Jacobian matrix. The linear periodically time dependent equation (3) may be decomposed into eigenfunctions according to the Floquet theory

$$\delta\boldsymbol{x}(t) = e^{(\lambda+i\omega)t}\boldsymbol{u}(t) \quad , \tag{4}$$

where the time periodic eigenfunction obeys

$$(\lambda + i\omega)\,\boldsymbol{u}(t) + \dot{\boldsymbol{u}}(t) = D_1\boldsymbol{f}(\boldsymbol{\xi}(t), 0)\boldsymbol{u}(t), \quad \boldsymbol{u}(t) = \boldsymbol{u}(t+T) \quad . \tag{5}$$

We remind the fact that the imaginary part of the exponent, ω, is defined modulo the driving frequency (cf. eq.(4)). Here the convention $\omega \in [0, 2\pi/T)$ will be used. The eigenvalue equation (5) admits as many solutions as the dimension of the equation of motion, but I refrain from labelling the different branches separately. The geometrical meaning of the Floquet exponents $\lambda + i\omega$ is quite simple. If one considers the periodic orbit $\boldsymbol{\xi}(t)$ as a closed curve in the phase space, then the real part λ respectively the imaginary part ω determine the radial expansion from respectively the revolution around the unstable orbit (cf. figure 1). Since the periodic orbit is unstable, at least one branch obeys $\lambda > 0$, and we consider these branches in what follows.

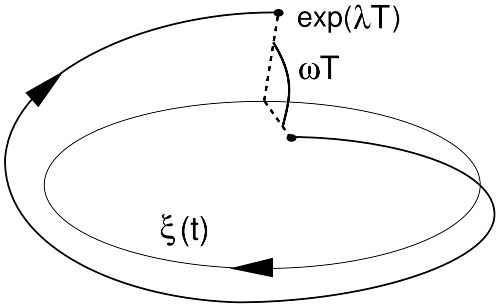

FIG. 1. Unstable periodic orbit $\boldsymbol{\xi}(t)$ and a neighbouring trajectory in the phase space. λ and ω denote the real and imaginary part of the Floquet exponent, T the period of the unstable orbit.

From the point of view of a simple control strategy one would like to take some difference between the actual state $\boldsymbol{x}(t)$ and the desired orbit $\boldsymbol{\xi}(t)$ to adjust a properly chosen control force. But for such an approach one needs the unstable periodic orbit in the phase space as well as some information about the equations of motion to couple the control force appropriately to the internal degrees of freedom. At least some phase space reconstruction techniques are necessary to achieve these goals. Whenever such a technique is not feasible, e. g. for the time scales or the number of degrees of freedom involved in the experiment, one has to resort to different methods, which should be easy to implement if the equations of motion are not known. Here the basic idea of delayed feedback control methods [7] enters. One just replaces the unknown periodic orbit in the above reasoning by retarding the trajectory by one period. When only a scalar signal is accessible by measurements one takes the difference

$$F(t) = g(\boldsymbol{x}(t)) - g(\boldsymbol{x}(t-\tau)) \tag{6}$$

as a control force. The coupling to the internal dynamics is realised by modulating one of the external parameters of the system with that force, e. g. the driving field or a different quantity. The whole setup is sketched in figure 2.

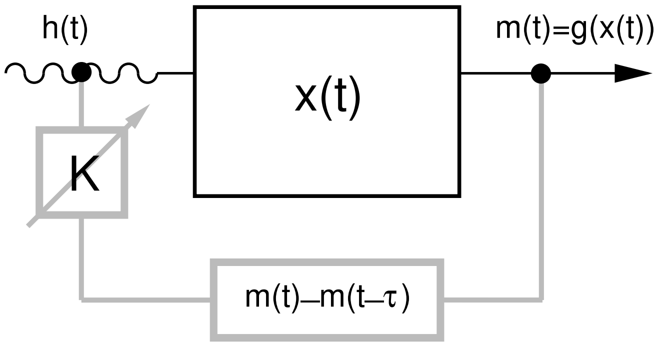

FIG. 2. Diagrammatic view of the experimental setup for delayed feedback control. The output signal $g(\boldsymbol{x}(t))$ is measured. The delay loop is displayed in grey. The delayed difference is used to modulate some system parameter, e. g. here the driving field. K denotes the amplitude of the feedback.

In order that such a control scheme works, the delay has to be adjusted to an integer multiple of the period T of the orbit. For the purpose of the following considerations I suppose that the period of the unstable orbit is known a priori and the delay has been fixed according to that value

$$\tau = T \quad . \tag{7}$$

Especially in explicitly time dependent driven systems such a condition usually does not impose any restriction since the periods of the orbits are integer multiples of the external driving period. Nevertheless, the problem how to estimate the period from the control signal in autonomous systems will be addressed later in section V.

Since one usually does not know how the control force couples to the internal degrees of freedom it is not clear by which mechanism the stabilisation of the unstable periodic orbit is achieved. In order to tackle this problem we have to investigate the system subjected to the control force, i. e. the quite general differential–difference equation given by eqs.(1) and (6). First of all we mention that the unstable periodic orbit $\boldsymbol{\xi}(t)$ is by construction a genuine orbit of the controlled system too. That statement is obvious (cf. eqs.(2) and (7)), since the control force (6) vanishes for

$\boldsymbol{x}(t) = \boldsymbol{\xi}(t)$ as long as the delay coincides with the period of the orbit. Hence the control scheme meets the constraint that it does not modify the orbit which we want to stabilise. In particular, the control signal $g(\boldsymbol{x}(t)) - g(\boldsymbol{x}(t-\tau))$ vanishes if successful control is achieved. The stability properties are again investigated by linear stability analysis. Proceeding as above one obtains for the deviations in linear order

$$\delta\dot{\boldsymbol{x}}(t) = D_1\boldsymbol{f}(\boldsymbol{\xi}(t),0)\delta\boldsymbol{x}(t) + Kd_2\boldsymbol{f}(\boldsymbol{\xi}(t),0)\left\{Dg(\boldsymbol{\xi}(t))\left[\delta\boldsymbol{x}(t) - \delta\boldsymbol{x}(t-\tau)\right]\right\} \quad , \tag{8}$$

where $d_2\boldsymbol{f}$ denotes the scalar derivative with respect to the second argument. Although we are now dealing with differential–difference equations we can apply the Floquet decomposition (4) [13]. Since the new exponents and eigenfunctions may depend on the control amplitude we use capital letters for the corresponding quantities and end up with the eigenvalue problem

$$(\Lambda + i\Omega)\,\boldsymbol{U}(t) + \dot{\boldsymbol{U}}(t) = M\left[K\left(1 - e^{-(\Lambda+i\Omega)\tau}\right)\right]\boldsymbol{U}(t), \quad \boldsymbol{U}(t) = \boldsymbol{U}(t+T) \quad , \tag{9}$$

where the abbreviation

$$M[\kappa]\boldsymbol{U} := D_1\boldsymbol{f}(\boldsymbol{\xi}(t),0)\boldsymbol{U} + \kappa d_2\boldsymbol{f}(\boldsymbol{\xi}(t),0)\left\{Dg(\boldsymbol{\xi}(t))\boldsymbol{U}\right\} \tag{10}$$

has been introduced for the matrix on the right hand side. It is important to note, that the control amplitude enters eq.(9) only through the parameter $\kappa = K\left(1 - \exp\left[-(\Lambda + i\Omega)\tau\right]\right)$. This feature is an immediate consequence of the choice of the control force (6). The stability properties, i. e. the Floquet exponents of the controlled orbit, $\Lambda + i\Omega$, are completely determined by the ordinary Floquet problem (9) for the matrix (10). Since eq.(9) constitutes an ordinary differential equation and the delay enters only implicitly via the mentioned parameter it can be handled very accurately if the model equations are known explicitly [14]. Here however we do not intend to resort to model equations but want to proceed without such a knowledge. Even then quite detailed and general information about the dependence of the Floquet exponents on the control amplitude is accessible.

The matrix (10) itself admits Floquet exponents which depend on the parameter κ. Let us denote this quantity by $\Gamma[\kappa]$. It depends continuously and even analytically on the parameter κ, as long as the Floquet exponents of (10) are non degenerated. Now, equation (9) tells us that the exponent of the controlled orbit $\Lambda + i\Omega$ obeys the constraint

$$\Lambda + i\Omega = \Gamma\left[K\left(1 - e^{-(\Lambda+i\Omega)\tau}\right)\right] \quad . \tag{11}$$

Of course, it is in general a difficult task to obtain a closed analytical expression for the quantity Γ, which then would depend on the system under consideration. Nevertheless we know that by definition (10) the boundary condition

$$\Gamma[0] = \lambda + i\omega \tag{12}$$

is fulfilled, since the matrix reduces to that of the uncontrolled system (cf. eq.(5)). One concludes that, whenever the real part of the exponent, Λ, changes its sign on the variation of the control amplitude, i. e. the orbit may become stable, then the corresponding frequency at that particular value of K has to be non vanishing, $\Omega \neq 0$. The reason for this fact is a quite simple consequence of eq.(11). Suppose on the contrary that at some value K both, the real as well as the imaginary part would vanish, i. e. $\Lambda + i\Omega = 0$. Then the left hand side of eq.(11) is zero, and the argument on the right hand side is zero too. But from the boundary condition (12) follows that the right hand side does not vanish, which yields a contradiction. Hence, whenever the real part of a Floquet exponent changes its sign, then the corresponding frequency is non vanishing. In some sense torsion is a necessary condition for an orbit to become stable. This feature is of pure geometric origin, and can be understood on the basis of figure 1 too. The control force is in a rough sense proportional to the difference $\boldsymbol{x}(t) - \boldsymbol{x}(t-\tau)$, i. e. to the distance between the endpoints of the trajectory shown in figure 1. But this distance would vanish if the orbit becomes stable without torsion. Thus one needs a finite frequency for stabilisation.

The above argument governs the change in sign of Λ for each branch of the eigenvalue spectrum. Whenever several eigenvalues are unstable, each real part has to become negative of course in order to achieve the stabilisation. In addition one should keep in mind that the condition for torsion yields a constraint for the frequency of the controlled orbit. At first sight it is not obvious how this constraint translates into the frequency of the uncontrolled orbit, since the control may alter the imaginary part of the exponents. Nevertheless frequencies $\omega = 0$ and $\omega = \pi/T$ are stable with respect to perturbations. A deviation of the frequency can be caused only, if a single Floquet multiplier collides with an additional multiplier giving rise to a complex conjugated pair, quite similar to eigenvalues of matrices. Combining this property with the argument of the preceding paragraph one concludes that unstable orbits without torsion, i. e. $\omega = 0$, cannot be stabilised by small control amplitudes K. In addition, orbits which have an odd number of positive exponents, i. e. $\Lambda > 0$ and $\Omega = 0$, cannot be stabilised at all, since a finite frequency is generated only from pairs of eigenvalues [15]. This result explains why for example delay methods fail to stabilise periodic orbits of the Lorenz model.

III. LIMITS OF THE SIMPLE FEEDBACK METHOD

Based on eq.(11) we are now going to discuss, for which values of the control amplitude stabilisation can be achieved. Let me stress that we have not employed any approximation yet. To get quantitative results let me resort to a Taylor series like expansion of eq.(11). If one takes the boundary condition (12) into account and applies a formal series expansion for small argument, one gets

$$\Lambda + i\Omega = \lambda + i\omega + (\chi' + i\chi'')K\left(1 - e^{-(\Lambda+i\Omega)\tau}\right) + \mathcal{O}(K^2) \quad . \tag{13}$$

Here the complex parameter $\chi' + i\chi'' := d\Gamma/d\kappa|_{\kappa=0}$ contains all the details of the system, i. e. the internal dynamics and the coupling of the control force to the degrees of freedom. Although expression (13) results from an expansion, it is not necessary that the control amplitude itself but only the argument in eq.(11) is a small quantity. Furthermore, it follows from the definition (10) that eq.(13) is exactly valid as long as the Jacobian matrix of the free dynamics, $D_1\boldsymbol{f}(\boldsymbol{\xi}(t),0)$, commutes with the control matrix $d_2\boldsymbol{f}(\boldsymbol{\xi}(t),0) \otimes Dg(\boldsymbol{\xi}(t))$. Altogether these arguments indicate that eq.(13) is a good starting point for quantitative results, where the parameters χ', χ'', λ, and ω may be fitted to actual data.

Equations of the type (13), in particular with real coefficients, typically occur if the stability of time independent stationary states of delay systems is investigated. In that context a whole theory for the solutions of such eigenvalue equations has been developed [16]. Within our context we do not treat the most general case but restrict for simplicity the subsequent analysis to unstable periodic orbits which flip their neighbourhood during one turn, i.e.

$$\omega = \pi/T \tag{14}$$

holds. Such orbits occur quite frequently in low dimensional dynamical systems. In fact, for three dimensional dissipative chaotic models, i. e. equations with a negative Jacobian, only orbits with frequency $\omega = 0$ and $\omega = \pi/T$ may be embedded into a chaotic attractor. Hence just the flipping orbits are accessible for delayed feedback control. Furthermore such orbits occur of course in the vicinity of period doubling bifurcations. Whenever $\omega = \pi/T$ holds, eq.(13) simplifies considerably. Since the Floquet multiplier of the unstable orbit is an isolated negative real number, $\exp[(\lambda + i\omega)T] = -\exp(\lambda T)$, the Floquet exponent of the matrix (10) is a real function, i. e. the Taylor series of $\Gamma[\kappa] - i\omega$ has real coefficients only. In particular $d\Gamma/d\kappa|_{\kappa=0}$ is real and χ'' vanishes. Therefore eq.(13) reads

$$\begin{aligned} \Lambda\tau &= \lambda\tau - (-\chi'\tau)K\left[1 + e^{-\Lambda\tau}\cos(\Delta\Omega\tau)\right] \\ \Delta\Omega\tau &= (-\chi'\tau)Ke^{-\Lambda\tau}\sin(\Delta\Omega\tau) \quad , \end{aligned} \tag{15}$$

where $\Delta\Omega = \Omega - \omega$ denotes the deviation of the frequency from the uncontrolled orbit. Eq.(15) has been written in terms of the dimensionless quantities $\Lambda\tau$, $\Delta\Omega\tau$, and $(-\chi'\tau)K$ for convenience. The parameter $\chi'\tau$, which incorporates the details of the system just fixes the scale for the control amplitude. First of all, for small values of K eq.(15) admits two solutions with a vanishing frequency deviation, $\Delta\Omega = 0$. One of them has a positive real part Λ and the other a negative one. The positive solution changes its sign at $K^{(\mathrm{fl})} = -\lambda/(2\chi')$, at least if the real part of Floquet exponent of the uncontrolled orbit, λ, is not too large. Hence stabilisation is achieved beyond this K–value. At even larger control amplitudes both real solutions collide and give rise to a pair of complex conjugate multipliers with a finite frequency deviation, $\Delta\Omega \neq 0$. Beyond that value the real part, Λ, increases again, since the influence of the control term in eq.(15) diminishes itself, if the frequency deviates from its optimal value $\Delta\Omega = 0$. In some sense the control amplitude is optimal at the frequency splitting point, since the stability of the controlled orbit is maximal. Figure 3 summarises these findings for a particular value of $\lambda\tau$.

In general one obtains a lower and an upper critical value for the control amplitude, which limit the interval for stabilisation. At the lower critical value the real part changes its sign with a frequency $\Omega = \pi/T$, i. e. a flip or period doubling bifurcation. At the upper critical value the frequency and its deviation is nonzero, which corresponds to a Hopf instability. Results which are in accordance with these theoretical predictions can be found in the literature. The dependence of the Floquet exponents on the control amplitude have been mostly obtained from numerical computations [7,17,18], where the exponents are accessible directly. But even data from electronic circuit experiments fit in within such a picture [12]. Here the exponents are estimated from the transient behaviour of the control signal $g(\boldsymbol{x}(t)) - g(\boldsymbol{x}(t-\tau))$ in the stable domain. The coincidence is not only qualitative but even quantitative, if one fits the quantities $\chi'\tau$ and $\lambda\tau$ to the data. Concerning the instability mechanism one typically observes a kind of reverse period doubling route at the lower critical control amplitude in accordance with the flip bifurcation predicted above. At the upper critical value incommensurate frequencies in the control signal are observed indicating the Hopf instability.

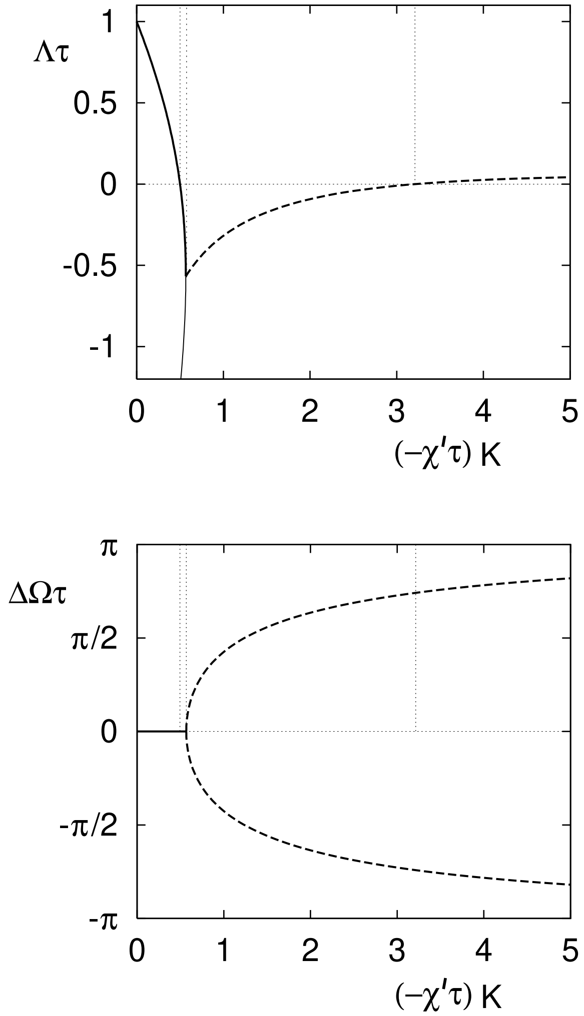

FIG. 3. Real part of the Floquet exponents and frequency deviations in dependence on the control amplitude. The thick solid line denotes the exponent with largest real part and the thick dashed line the complex conjugated pair. The thin solid line indicates the lower lying real exponent. The (minimal and maximal) critical and the optimal control amplitudes are indicated by thin vertical dashed lines. The curves have been obtained from eq.(15) with $\lambda\tau = 1$.

Inspecting figure 3 one argues that stabilisation is not possible if $\lambda\tau$ is too large, since the branching of eigenvalues then occurs for positive real part. To clarify this point let us study the critical control amplitudes in dependence on $\lambda\tau$. For the lower critical value, i. e. the flip instability, $\Lambda = 0$, $\Delta\Omega = 0$, we obtain from eq.(15)

$$(-\chi'\tau)K^{(\mathrm{fl})} = \frac{\lambda\tau}{2} \quad . \tag{16}$$

On the other hand, for the Hopf instability, $\Lambda = 0$, $\Delta\Omega \neq 0$ eq.(15) yields the critical curve in a parametric representation

$$\begin{aligned} (-\chi'\tau)K^{(\mathrm{ho})} &= \frac{\Delta\Omega\tau}{\sin(\Delta\Omega\tau)} \\ \lambda\tau &= \frac{\Delta\Omega\tau}{\tan(\Delta\Omega\tau/2)} \quad . \end{aligned} \tag{17}$$

Both boundaries are displayed in figure 4 together with the region in the K–$\lambda\tau$ parameter plane where stabilisation works successfully.

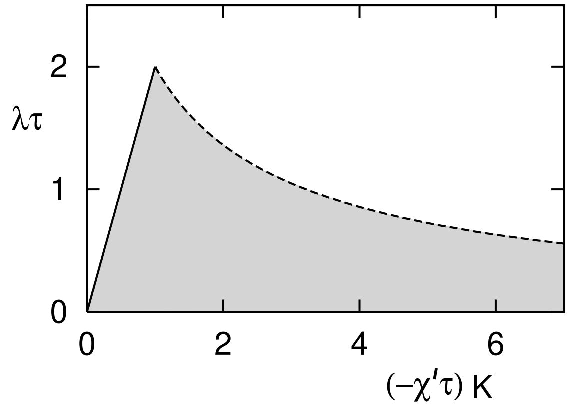

FIG. 4. Stability domain for simple delayed feedback control. Solid/dashed lines indicate the domain boundary caused by flip/Hopf bifurcations (cf. eq.(16)/(17)). The frequency deviation on the Hopf line obeys $\Delta\Omega\tau \in [-\pi, \pi]$.

It is obvious from eqs.(16) and (17) that stabilisation is not possible for $\lambda\tau \geq 2$, since the generation of a finite frequency deviation already occurs for $\Lambda > 0$. This property explains the common numerical observation that unstable periodic orbits with large Floquet exponents or long periods cannot be stabilised by the simple feedback method. The critical value $\lambda\tau = 2$ does not depend on the system parameter χ' within our approximation. However, one cannot expect this value to be universal, since higher order terms to eq.(13), which have been neglected in our approach contribute to the numerical value. Then the critical value will depend on the system and even the orbit under consideration. Nevertheless, the estimate of the critical value coincides quite reasonably with numerical simulations, which can be found in the literature (cf. e. g. [17]). In addition, the features shown in figure 4 have also been observed in numerical simulations and even in real experiments [19]. In such realisations unstable periodic orbits with different Floquet exponents are generated by the variation of an external system parameter, and one monitors the stability domain in dependence on the external parameter. Since the unstable exponent of the periodic orbit does not depend in a simple way on such parameters, and since the scaling parameter χ' may vary, one observes a slightly distorted version of the image shown in figure 4. However, at least a good qualitative coincidence is obtained (cf. [19]).

Altogether the whole analysis shows that main features of the control mechanism are reproduced quantitatively already by the low order approximation of the characteristic equation (11). Therefore the findings presented here occur in some sense typically for time delayed feedback control.

IV. ADVANCED CONTROL STRATEGIES

As demonstrated in the last section one severe restriction for delayed feedback control is caused by the size of the largest Floquet exponent or the length of the period of the orbit. Inspecting the result depicted in figure 3 the constraint originates, at least for flipping orbits, from the frequency splitting at finite control amplitude. It has to be our goal to shift this splitting to larger values of K.

One might argue to avoid the mentioned restrictions by an appropriate coupling of the control force to the internal degrees of freedom. Beside the fact that such an approach calls for a detailed knowledge about the internal dynamics and about the precise mechanism which couples the external force to the degrees of freedom, such a strategy cannot cure the restrictions mentioned above in general. The formulation of the preceding section was held general enough to incorporate all possible functional dependencies of the force (cf. eq.(1)), e. g. even a periodic modulation of the control amplitude. As long as the low order approximation (13) is valid, such modifications only determine the scale for the control amplitude K. Hence one has to modify the control force (6). However, the constraint that the orbit of the system is not altered has to be observed. Altogether a quite general expression which fits in within such a constraint is given by

$$F(t) = \int_0^\infty \Phi(t') \left[g(\boldsymbol{x}(t-t')) - g(\boldsymbol{x}(t-\tau-t')) \right] dt' \quad . \tag{18}$$

Eq. (18) contains the simple control scheme (6) via the choice $\Phi(t) = \delta(t-0)$. Much more general schemes can be realised of course. E. g. a simple low pass filtering of the control signal in the setup of figure 2 corresponds to an exponentially decaying integral kernel $\Phi(t)$. Expression (18) seems to contain all possible choices which are accessible in experimental situations. In what follows let me concentrate on cases where only integer multiples of the period $T = \tau$ enter the expression, i. e. the kernel consists of a finite or infinite sum of δ–contributions. Then eq.(18) reads

$$F(t) = \sum_{\nu=0}^{\infty} s_\nu \left[g(\boldsymbol{x}(t-\nu\tau)) - g(\boldsymbol{x}(t-(\nu+1)\tau)) \right] \quad , \tag{19}$$

where we suppose that the sum $\sum s_\nu$ converges absolutely in order to keep the expression well defined. Control forces of such a structure, especially with a geometrically decreasing series $\{s_\nu\}$, have been suggested as good candidates to circumvent the limitations mentioned above [20]. Numerical analysis of model equations confirm this point of view [18,14]. In addition, I focus here on control forces of the form (19) since the analysis is a little bit simpler. Nevertheless, the general case (18) can be treated also by a slightly different approach.

The stability properties of the control scheme (1), (7), and (19) are studied along the lines of the preceding section. The linear stability analysis yields a Floquet problem where the exponents of the controlled orbit obey the eigenvalue equation

$$(\Lambda + i\Omega)\, \boldsymbol{U}(t) + \dot{\boldsymbol{U}} = M \left[K \left(1 - e^{-(\Lambda+i\Omega)\tau}\right) \Sigma \left(e^{-(\Lambda+i\Omega)\tau}\right)\right] \boldsymbol{U}(t), \quad \boldsymbol{U}(t) = \boldsymbol{U}(t+T) \quad . \tag{20}$$

The matrix M is given by the definition (10) and Σ denotes a kind of Z–transform. i. e. a discrete Laplace transform of the series $\{s_\nu\}$

$$\Sigma(z) = \sum_{\nu=0}^{\infty} s_\nu z^\nu \quad . \tag{21}$$

We ended up with an ordinary Floquet problem since the control force (19) contains integer multiples of the period. In the general case (18) one would have obtained a delay Floquet problem which is a little bit harder to treat. The sole difference to eq.(9) comes through the argument of the matrix M. As already explained in the preceding section eq.(20) implies an implicit relation for the Floquet exponent of the controlled orbit in terms of the Floquet exponent Γ of the matrix (10)

$$\Lambda + i\Omega = \Gamma \left[K \left(1 - e^{-(\Lambda+i\Omega)\tau}\right) \Sigma \left(e^{-(\Lambda+i\Omega)\tau}\right)\right] \quad . \tag{22}$$

The difference to the case (11), which governs the simple control scheme is just the argument appearing on the right hand side of eq.(22). Hence the conclusions of the preceding section about the torsion of the orbits are also valid for the advanced control schemes.

For the quantitative evaluation we again resort to a first order Taylor series expansion of Γ. Since the focus is on the improvement of the control scheme I also specialise to the case of the flipping orbit, $\omega = \pi/T$, which was extensively analysed in the preceding section from the point of view of the simple control force. Using the notation of eq.(15) one obtains an implicit equation for the Floquet exponents of the controlled orbit

$$\Lambda\tau + i\Delta\Omega\tau = \lambda\tau - (-\chi'\tau) K \left(1 + e^{-(\Lambda+i\Delta\Omega)\tau}\right) \Sigma \left(-e^{-(\Lambda+i\Delta\Omega)\tau}\right) \quad . \tag{23}$$

From this expression we can evaluate the dependence of the exponents on the control parameters, i. e. on the control amplitude K and the relative weights $\{s_\nu\}$ of the different control terms. Of course a kind of normalisation is required, since one of the parameters K, $\{s_\nu\}$ is superfluous. As there is no natural prescription obvious, I resort henceforth to the condition $s_0 = 1$ which has been used so far in the literature. Of special interest is the region in the parameter space where control is possible. The boundary of this region is given by the condition that the real part Λ vanishes and the frequency deviation $\Delta\Omega$ is either zero or nonzero. For the first case $\Delta\Omega \equiv 0$, i. e. a boundary caused by a flip bifurcation, eq.(23) yields (cf. eq.(16))

$$(-\chi'\tau) K^{(\mathrm{fl})} = \frac{\lambda\tau}{2\Sigma(-1)} = \frac{\lambda\tau}{T(0)} \quad . \tag{24}$$

Here we have already introduced the abbreviation

$$T(\phi) = \left(1 + e^{-i\phi}\right) \Sigma\left(-e^{-i\phi}\right) \tag{25}$$

which is sometimes called a transfer function of the force (19), since it determines the linear response of the force $F(t)$ with respect to the scalar quantity g. It is worth to mention that the result (24) is valid beyond the approximation (23), since it can be obtained from the full equation (22), if one replaces the value of $\lambda\tau/(-\chi'\tau)$ by the real solution κ of $\Gamma[\kappa] = i\pi/\tau$. For the second case $\Delta\Omega \neq 0$, i. e. a boundary caused by a Hopf instability, eq.(23) results in a parametric representation (cf. eq.(17))

$$(-\chi'\tau)K^{(\mathrm{ho})} = -\frac{\Delta\Omega\tau}{\mathrm{Im}T(\Delta\Omega\tau)}$$
$$\lambda\tau = -\frac{\Delta\Omega\tau \mathrm{Re}T(\Delta\Omega\tau)}{\mathrm{Im}T(\Delta\Omega\tau)} \quad . \tag{26}$$

For further discussions of the stability domains one requires properties of the transfer function (25), i. e. of the Z–transform. Therefore let me switch to an important example where the relative weights $\{s_\nu\}$ constitute a geometrically decreasing sequence

$$s_\nu := R^\nu, \quad |R| < 1 \quad . \tag{27}$$

This scheme was originally introduced in [20] and it is quite easy to implement in real experimental setups. With the choice (27) the Z–transform (21) results in a simple geometric series and the evaluation of the transfer function (25) is straightforward. Finally the boundaries of the stability domain of a flip orbit are obtained from eq.(24) as

$$(-\chi'\tau)K^{(\mathrm{fl})} = \frac{\lambda\tau}{2}(1+R) \tag{28}$$

and from eq.(26) as

$$(-\chi'\tau)K^{(\mathrm{ho})} = \frac{\Delta\Omega\tau}{2\tan(\Delta\Omega\tau/2)}\left[\left(\frac{1+R}{1-R}\right)^2 + \tan^2\frac{\Delta\Omega\tau}{2}\right](1-R)$$
$$\lambda\tau = \frac{\Delta\Omega\tau}{\tan(\Delta\Omega\tau/2)}\frac{1+R}{1-R} \quad . \tag{29}$$

Figure 5 contains these boundaries for several values of R.

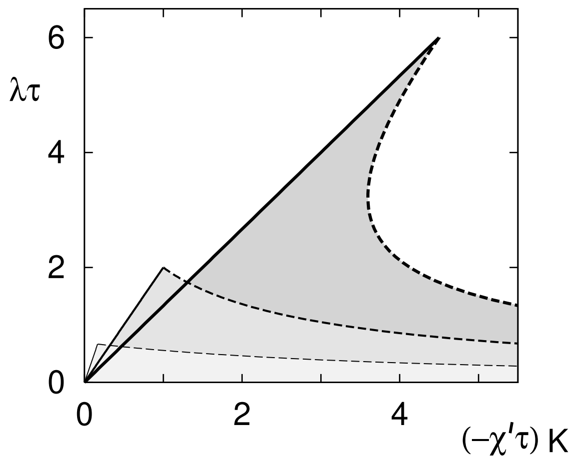

FIG. 5. Stability domain for extended delayed feedback control (27) for different values of R. Solid/dashed lines indicate the domain boundary caused by the flip/Hopf bifurcation (cf. eq.(28)/(29)). Thick, medium, and thin lines correspond to $R = 0.5$, $R = 0.0$, and $R = -0.5$. The frequency on the Hopf lines obeys $\Delta\Omega\tau \in [-\pi, \pi]$.

The stability domains extend up to a maximal exponent $\lambda\tau = 2(1+R)/(1-R)$. Hence more orbits can be stabilised with this extended control scheme for positive values of R. In addition figure 5 coincides at least qualitatively with numerical simulations and data from electronic circuit experiments [19]. The coincidence is even quantitative at least for small values of $\lambda\tau$. Of course, the simple first order Taylor series approximation, which has been employed to obtain the result (28) and (29) may fail quantitatively for large values of $\lambda\tau$. Nevertheless, the above considerations explain, why the extended control scheme improves the simple delay method considerably.

The limiting case $R \uparrow 1$ of the preceding paragraph already indicates that a delay force with long memory favours the stabilisation of orbits with large values of $\lambda\tau$. Such a quite general statement can be obtained also on the basis of the expressions (24) and (26). However, control forces with a long memory seem to be difficult to realise in experimental situations. Furthermore a sequence of weights $\{s_\nu\}$ with alternating signs seems to decrease the stabilisation of flip orbits, in rough correspondence with what is expected from eq.(19).

V. INFLUENCE OF A DELAY MISMATCH

For the preceding analysis we have presupposed, that the period of the unstable orbit, T, is known a priori. Thus, the delay can be chosen accordingly (cf. eq.(7)). Whenever such periods are not fixed by an external time scale, e. g. by the period of a driving field, then the proper choice of the delay may become a considerable problem. In fact, if the delay does not coincide with an integer multiple of the true period, then the system subjected to control performs its own complicated nonlinear motion but can never reach the desired orbit $\boldsymbol{\xi}$. Hence, one requires a recipe, how to extract the period of the genuine orbit from the control signal. The corresponding problem will be discussed in the context of the simple control scheme (6). The considerations for advanced control schemes are quite similar and do not deserve a separate discussion. Furthermore, autonomous systems will be considered in what follows, since otherwise the determination of the period becomes quite trivial.

Empirical and semi–empirical schemes have been developed to extract the proper period from the measurement of the scalar quantity or the control signal $g(\boldsymbol{x}(t)) - g(\boldsymbol{x}(t-\tau))$. The basic idea is quite simple. One looks for some kind of periodicity in such signals and tries to change the delay accordingly. The first attempt in this direction just monitors the aperiodic control signal and extracts an estimate for the period from successive maxima [21]. A refined method supplements this search by minimising an average of the control signal [22] and employing ideas from adaptive control processes [1]. Altogether these schemes have turned out to work quite well in numerical simulations of autonomous differential equations, but a deeper theoretical explanation is missing.

Here I want to follow a slightly different approach. A theoretical and systematic analysis will be presented, which is based on the general equations of motion (1) and (6) without taking the explicit functional form of these expressions into account. Of course, one cannot expect to obtain results for arbitrary delay time, since the dynamics might become model dependent. However, if the delay time τ is in some sense close to the desired period T one can succeed with such a theoretical approach. We know by presupposition that the system subjected to the control force has the periodic solution $\boldsymbol{x}(t) = \boldsymbol{\xi}(t) = \boldsymbol{\xi}(t+T)$ provided that the delay equals the period, $T = \tau$. Whenever the deviation between the period and the delay is not too large this orbit changes only slightly, and this statement is even rigorous under quite general mathematical conditions [13]. To be definite, for $\tau \neq T$ this new fictitious periodic orbit, $\boldsymbol{\eta}$, has a shape and a period Θ which depend on the control amplitude K and the delay time τ, i. e. $\Theta = \Theta(K, \tau)$ and $\boldsymbol{\eta} = \boldsymbol{\eta}_{K,\tau}$. It is not a genuine orbit of the uncontrolled system, but tends continuously to the true period orbit $\boldsymbol{\xi}$ in the limit $\tau \to T$. Then of course

$$\Theta(K, \tau = T) = T \tag{30}$$

holds. Furthermore, the fictitious orbit inherits its stability properties from the true orbit, i. e. $\boldsymbol{\eta}$ is stable whenever $\boldsymbol{\xi}$ is stable under control and vice versa. Nevertheless, even if the orbit $\boldsymbol{\eta}$ is stable it does not yield a genuine orbit of the uncontrolled system, since the control signal does not vanish.

The strategy to adapt the delay is quite simple. Supposes that one observes a periodic but non vanishing control signal, i. e. the control amplitude is fixed so that the fictitious orbit is stable. Then the period Θ is accessible to the experimental observation. Since the control amplitude K and the delay are adjustable parameters one can measure directly the function $\Theta(K, \tau)$ in certain ranges of the control amplitude K and the delay time τ. The desired period we are looking for just obeys eq.(30). Hence the problem to determine the desired period is reduced to the root finding of a function which can be determined from the measured signal. Figure 6 contains a diagrammatic view of this construction.

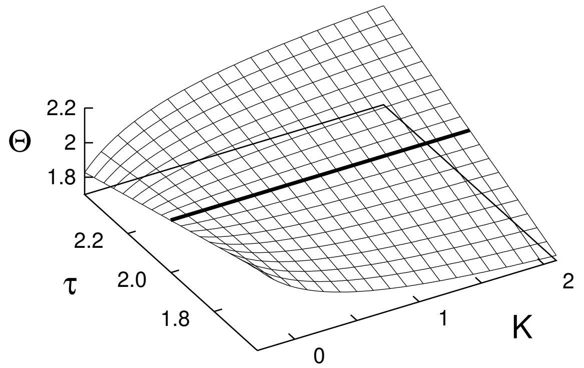

FIG. 6. Diagrammatic view of the fictitious period Θ in dependence on the control amplitude K and the delay time τ. For $\tau = T$ the fictitious period equals the true one T (solid line). The actual graph has been obtained from the analytical expression (37) with $T = 2.0$ and $\alpha = -1.0$.

Beyond such an elementary consideration an analytical expression for the fictitious period Θ would help to implement the above mentioned scheme. To address the problem how to calculate this function from the general equations of motion, let me first employ a simple Taylor series expansion with respect to the deviation of the delay from the period, by taking the boundary condition (30) into account

$$\Theta(K,\tau) = T + \partial_\tau \Theta|_{\tau=T}(\tau - T) + \mathcal{O}\left((\tau-T)^2\right) \quad . \tag{31}$$

Of course the coefficients of this expansion depend on the control amplitude. This dependence will be evaluated quantitatively from eqs.(1) and (6).

To achieve this goal, let me rewrite the equation of motion for the periodic orbit $\boldsymbol{\eta}$ in terms of the dimensionless time $s := t/\Theta$, $\bar{\boldsymbol{\eta}}(s) = \boldsymbol{\eta}(\Theta s)$

$$\bar{\boldsymbol{\eta}}'(s) = \Theta \boldsymbol{f}(\bar{\boldsymbol{\eta}}(s), K(g[\bar{\boldsymbol{\eta}}(s)] - g[\bar{\boldsymbol{\eta}}(s-\tau/\Theta)])), \quad \bar{\boldsymbol{\eta}}(s) = \bar{\boldsymbol{\eta}}(s+1) \quad . \tag{32}$$

The boundary value problem (32) determines in principle the fictitious orbit $\bar{\boldsymbol{\eta}}$ as well as the period Θ. Since we are interested in the derivative $\partial_\tau \Theta|_{\tau=T}$ we differentiate with respect to τ, taking into account that $\bar{\boldsymbol{\eta}}$ depends explicitly on the delay

$$\begin{aligned}
&(\partial_\tau \bar{\boldsymbol{\eta}})' - \Theta D_1 \boldsymbol{f}(\ldots)\partial_\tau \bar{\boldsymbol{\eta}}(s) - \Theta K d_2 \boldsymbol{f}(\ldots)\left\{dg[\bar{\boldsymbol{\eta}}(s)]\partial_\tau \bar{\boldsymbol{\eta}}(s) - dg[\bar{\boldsymbol{\eta}}(s-\tau/\Theta)]\partial_\tau \bar{\boldsymbol{\eta}}(s-\tau/\Theta)\right\} \\
&= (\partial_\tau \Theta)\boldsymbol{f}(\ldots) + \Theta K \partial_\tau(\tau/\Theta) d_2 \boldsymbol{f}(\ldots)\left\{dg[\bar{\boldsymbol{\eta}}(s-\tau/\Theta)]\bar{\boldsymbol{\eta}}'(s-\tau/\Theta)\right\} \quad .
\end{aligned} \tag{33}$$

Here the arguments of $\boldsymbol{f}$, which are just indicated by $\ldots$, coincide with those from eq.(32). The terms involving the derivative of the orbit with respect to the explicit τ–dependence, $\partial_\tau \bar{\boldsymbol{\eta}}$, have been collected on the left hand side. In order to isolate the quantity of interest, $\partial_\tau \Theta$, we trace back to some kind of Fredholm condition. If one considers the left hand side of eq.(33) as a linear operator acting on the derivative $\partial_\tau \bar{\boldsymbol{\eta}}$, then this operator admits a null–eigenfunction. It is related to the time translation invariance of the original system. In fact, taking the derivative of eq.(32) with respect to s yields

$$0 = (\bar{\boldsymbol{\eta}}')' - \Theta D_1 \boldsymbol{f}(\ldots)\bar{\boldsymbol{\eta}}'(s) - \Theta K d_2 \boldsymbol{f}(\ldots)\left\{dg[\bar{\boldsymbol{\eta}}(s)]\bar{\boldsymbol{\eta}}'(s) - dg[\bar{\boldsymbol{\eta}}(s-\tau/\Theta)]\bar{\boldsymbol{\eta}}'(s-\tau/\Theta)\right\} \quad . \tag{34}$$

Hence $\bar{\boldsymbol{\eta}}'$ gives the above mentioned right–null–eigenfunction. $\bar{\boldsymbol{\zeta}}(s)$ should denote the corresponding left–null–eigenfunction with respect to the canonical bilinear form $\int_0^1 \boldsymbol{v}(s)\boldsymbol{u}(s)\,ds$. When eq.(33) is multiplied with this eigenfunction all the terms on the left hand side involving $\partial_\tau \bar{\boldsymbol{\eta}}$ vanish. Therefore we are left with

$$0 = \partial_\tau \Theta \int_0^1 \bar{\boldsymbol{\zeta}}(s)\boldsymbol{f}(\ldots)\,ds + \Theta K \partial_\tau(\tau/\Theta)\int_0^1 \bar{\boldsymbol{\zeta}}(s) d_2 \boldsymbol{f}(\ldots)\left\{dg[\bar{\boldsymbol{\eta}}(s-\tau/\Theta)]\bar{\boldsymbol{\eta}}'(s-\tau/\Theta)\right\}\,ds \quad . \tag{35}$$

The details of the system are solely contained in the integrals and have been condensed to simple numbers. But in general the integrals depend on the delay τ and in particular on the control amplitude K through the left–null–eigenfunction $\bar{\boldsymbol{\zeta}}$ (cf. eq.(34)). However, if we evaluate all expressions at $T = \tau$ then $\Theta = T = \tau$ holds, and the delay in the argument of $\bar{\boldsymbol{\eta}}$ drops by virtue of the boundary condition (cf. eq.(32)). By the same argument the linear operator in eq.(34) and hence the eigenfunction $\bar{\boldsymbol{\zeta}}$ becomes independent of K. Hence, the integrals in eq.(35) become constant real numbers for $T = \tau$ and we finally obtain

$$0 = \alpha \partial_\tau \Theta|_{\tau=T} + TK\partial_\tau(\tau/\Theta)|_{\tau=T} \quad . \tag{36}$$

Here α denotes the ratio of the integrals occurring in eq.(35). This single parameter contains all the details of the system under consideration. Let me stress that eq.(6) has fixed the K–dependence of the Taylor series coefficient $\partial_\tau \Theta|_{\tau=T}$. If we solve for this quantity and truncate the expansion (31) at the first order we end up with

$$\Theta(K,\tau) = T + \frac{K}{K-\alpha}(\tau - T) \quad . \tag{37}$$

The explicit expression (37) relates the measurable fictitious period Θ with the control amplitude K, the delay time τ and the unknown true period T. Its use is straightforward. One just monitors the K–dependence of the fictitious period and then estimates the true period. Already two measurements, i. e. two values of K and τ at which a periodic control signal is visible, are sufficient to solve for the desired period T and the system depended parameter α. In fact, the approach works well for numerical simulations of autonomous differential equations [23], even for quite large deviations $T-\tau$. The spirit of the formula (37) is similar to a Newton scheme for the root finding procedure mentioned above. One should however be aware, that by changing the delay or the control amplitude a former periodic state may become unstable. Hence, it may be necessary to readjust the control amplitude appropriately if one changes the delay according to eq.(37). It is possible to develop an adaptive control process for dealing with this complication, but its presentation is beyond the scope of this article. Although we have presupposed to observe a periodic control signal, the approach seems to work at least in numerical simulations, if an almost periodic signal, i. e. a signal with a dominant peak in the spectrum, is observed. From that point of view the approach presented here explains to some extent the success of the empirical schemes described above.

VI. SUMMARY

The present article has demonstrated that a quite general and analytical theory for delayed feedback control techniques can be developed. It was shown that the success of control of periodic orbits depends on the torsion of neighbouring trajectories, i. e. a purely geometrical property in the phase space. Beyond such a qualitative statement, a quantitative evaluation was possible by supplementing the linear stability analysis with a simple Taylor series approximation. Here the peculiarities of the system under consideration have been condensed to the Taylor coefficients which may be regarded as fit parameters. Despite its simplicity the approach works quite well and explains the bifurcations by which stabilisation is achieved, as well as the domains in parameter space where control works successfully. The empirically well known limit of delayed feedback methods to orbits with short periods or small Lyapunov exponents is explained by a frequency splitting of eigenvalues giving rise to an optimal value of the control amplitude. Advanced strategies, which involve multiple integers of the delay time may overcome this limitation. The application of delayed feedback methods for controlling periodic orbits requires the adjustment of the delay to an integer multiple of the period. Such periods, if unknown a priori, can be obtained from the control signal, since the true periodic orbit induces a fictitious orbit in the full delay system. Based on this observation a strategy was described, in order to extract the true period from a periodic component of the control signal. An expression was derived which relates the period of the control signal with the true period, the control amplitude, and the delay time. The above mentioned features have been extensively discussed within the example of flipping orbits. Although these orbits occur quite frequently in low dimensional dissipative systems, the whole approach is clearly not limited to this case.

From the theoretical point of view I have employed simple perturbation schemes, which can deal with quite general systems. All the peculiarities of the concrete experimental situation, especially the influence of the coupling of the control force to the internal degrees of freedom, are condensed to a few real numbers. Of course, as a drawback the predictions may fail quantitatively and even quantitatively in concrete situations, whenever the simple Taylor series like approximations break down. But then, I suppose that all the details depend strongly on the system under consideration and deserve in each case a separate discussion.

The implementation of delayed feedback methods turns the original system into a delay–differential equation. However, this property does not cause any essential complication, since the stability is again determined by an ordinary Floquet equation, which already governs stability properties of periodic orbits in ordinary differential equations. Hence, our incomplete understanding of delayed feedback control originates from the lack of an analytical theory of Floquet problems. I believe that in this sense the potential infinite dimensional phase space of delay systems is far less important for delayed feedback control, in contrast to some statements in the literature.

I have completely focussed on the linear stability analysis. All properties related to some nonlinear analysis, e. g. the domain of attraction of controlled periodic orbits in the phase space, are expected to depend on the system under consideration, and seem to require a proper theoretical modelling in each case.

Finally I would like to stress that the number of degrees of freedom in the uncontrolled system does not play a very significant role in the approach presented here. Nevertheless, one would expect novel features, if the limit of large number of degrees of freedom would be considered, i. e. the case of space time chaos control in systems of large aspect ratio. The latter problem seems to be the most challenging for future investigations in the field of chaos control.

Altogether, not only the results presented here but in particular the type of approach chosen for the investigation should stimulate the understanding of the conceptually simple delayed feedback control techniques.

ACKNOWLEDGEMENT

Almost all ideas which have been presented here have developed from many discussions with H. Benner, E. Reibold, J. Hołyst, and T. Bernard.

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