Frequencies of Bogoliubov coupled oscillators with resonant three particle interactions

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Abstract. Oscillators $\sum \omega_k a_k^\dagger a_k$ with interactions $h/2 \sum V_k a_k^\dagger a_{-k}^\dagger$ + c.c. and $\sum V_{k_1,k_2\,k_3} a_{k_1}^\dagger a_{k_2} a_{k_3}$+c.c. are discussed as a function of a threshold parameter h, where it is especially investigated to take resonant terms $V_{k_1,k_2\,k_3}$ with $\omega_{k_1} - \omega_{k_2} - \omega_{k_3} \approx 0$ properly into account. A unitary transformation is formulated to transform the Hamiltonian into $E_0 + \sum \hat{\omega}_k(h) a_k^\dagger a_k + \cdots$, where the remaining interactions do not contain products of creation operators or annihilation operators only and vanish for $V_{k_1,k_2\,k_3} \rightarrow 0$. The new frequencies $\hat{\omega}_k(h)$ are evaluated for weak three particle couplings in terms of the parameters of the Hamiltonian. As expected, the frequencies $\hat{\omega}_k(h)$ are renormalised by damping rates and frequency shifts in the case of small spacings of the wave vectors, and the threshold of h for $\hat{\omega}_k(h)$ to break down, is essentially changed by the damping rates.

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1. Introduction

Driven dissipative systems are still one of the fascinating topics in statistical physics. Systems subjected to nonequilibrium conditions show phenomena like pattern formation, instabilities and chaotic motion [1]. Quite often and very successfully one describes these phenomena on a phenomenological level, e.g. by hydrodynamic or transport equations. One should however keep in mind that fully systematic concepts based on first principles are still missing , apart from approaches using assumptions like linear response or local equilibrium.

Such missing links become obvious when one considers strongly driven magnetic systems. It is known for decades that instabilities occur under strong driving [2] but the type of motion beyond the instability is still unclear [3, 1, 4, 5, 6]. There exist phenomenological approaches for treating spin wave instabilities [7]. But these concepts do not take damping in a systematic fashion into account.

In fact, nonequilibrium statistical operators for thermodynamic systems may be obtained from first principles directly from the driven Liouville equation [8, 9]. The price one has to pay consists in the development of suitable perturbation schemes which treat driving and damping in a systematic way. We are going to address this question here. Since we are interested in the main features of such a perturbation expansion we focus to a large extent on static fields so that the approach can be reduced to the investigation of effective Hamiltonians. Such a limitation is however often not severe since the time dependent case can be treated frequently in a similar fashion by applying some version of averaging [10].

Motivated by the physics of ferromagnetic magnons, we consider a system of oscillators which are coupled by interactions $h/2\sum V_k a_k^\dagger a_{-k}^\dagger + \mathrm{c.c.}$ and $\sum V_{k_1,k_2\,k_3} a_{k_1}^\dagger a_{k_2} a_{k_3} + \mathrm{c.c.} + \cdots$. In the magnetic case, these types of interaction arise from the dipole dipole interaction and external magnetic fields [11, 4], however we will not apply to a specific realization of the parameters in our Hamiltonian. The coupling h is considered to be the external variable of interest.

For $V_{k_1,k_2\,k_3} = 0$ and vanishing higher interactions, it is well known [12] that one can use a Bogoliubov transformation for the creation and annihilation operators or a corresponding unitary transformation of the Hamiltonian, to obtain decoupled oscillators with frequencies $\sqrt{\omega_k^2 - h^2|V_k|^2}$. Such an approach implies that the interaction strength is restricted to the range $h^2 < \min\{\omega_k^2/|V_k|^2\}$. Thus, if this minimum is extremely small, neglected interactions will be important. From dynamic equations of expectation values for a_k and $a_k^\dagger$ one can see, that these remaining interactions essentially enter into the threshold of h by damping rates increasing the bare value [7]. This point is essential, if $\min\{\omega_k^2/|V_k|^2\} = 0$, so that without damping there is no range of stability.

The aim of this paper is to present a static§ treatment of the Hamiltonian, along the lines without three particle interactions, as to provide an approach to the threshold

§ In this context the terminus static will be used for time independent unitary transformations, whatever the way, they are calculated.

problem for h in the interacting case, without setting up dissipative equations of motion for expectation values. In particular we investigate, how damping rates can appear in the parameters of a transformed Hamiltonian. This can be considered as a first step to find a static access also in the range of h beyond the modified threshold, although one cannot expect, that such an attempt will be less complex than discussing equivalent nonlinear dynamic equations.

An idea for a static treatment of the Hamiltonian would be, to perform the standard Bogoliubov transformation to new creation and annihilation operators, $\hat{a}_k^\dagger = u_k(h)\, a_k^\dagger + v_k(h)\, a_{-k}$, and to handle the transformed interaction terms by perturbation theory. This procedure, however, is not adequate, as without carrying out infinite summations, h is restricted to values below the bare threshold. Furthermore, when h approaches the bare threshold, the parameters $u_k(h)$ and $v_k(h)$ of the critical wave vectors tend to infinity, so that there are transformed interaction coefficients which diverge. On the other hand a generalization of the Bogoliubov transformation with $a_k^\dagger$ as an infinite series of products of new creation and annihilation operators or vice versa, will be extremely difficult. This is why we choose an alternative route, and start with a unitary transformation of the original Hamiltonian, so that the terms $a_k^\dagger a_{-k}^\dagger$ + c.c. are removed, but higher orders of the interactions are included in the unitary transformation from the beginning.

In section 2 we define the desired unitary transformation and show, how the new frequencies $\hat{\omega}_k$ can be found, whereas in section 3 we evaluate these frequencies for weak $V_{k_1,k_2\,k_3}$ by using correlation functions or their Markovian approximations respectively. The main results of the perturbation expansion are summarised in section 4. Our purely algebraic approach presented here is so flexible that it can be applied immediately to dynamical models under strong equilibrium conditions. In the outlook, section 5, we will show that the presented technique will be useful for treating the steady states of a periodically driven ferromagnet. Thus our approach resembles to some extent averaging procedures which have proven to be fruitful in nonlinear dynamics [10].

2. Formal theory

We consider a bosonic Hamiltonian

$$\mathcal{H} = \mathcal{H}_{harm} + \mathcal{H}_{in} + h\mathcal{H}^{(1)} = \mathcal{H}^{(0)} + h\mathcal{H}^{(1)}, \tag{1}$$

where

$$\mathcal{H}_{harm} = \hbar \sum_k \omega_k\, a_k^\dagger a_k \tag{2}$$

$$\mathcal{H}_{in} = \hbar \sum_{k_1\,k_2\,k_3} (V_{k_1,k_2\,k_3} a_{k_1}^\dagger a_{k_2} a_{k_3} + V_{k_1,k_2\,k_3}^* a_{k_2}^\dagger a_{k_3}^\dagger a_{k_1}) + \cdots \tag{3}$$

$$\mathcal{H}^{(1)} = \hbar/2 (\sum_k V_k\, a_k^\dagger a_{-k}^\dagger + V_k^*\, a_k a_{-k}). \tag{4}$$

Translation invariance is assumed which means that the coefficients $V_{k_1,k_2\,k_3}$ vanish unless $k_1 - k_2 - k_3 = 0$. The dots in (3) indicate higher order interactions which preserve the vacuum state as an eigenvector and ensure a discrete energy spectrum of $\mathcal{H}^{(0)}$. The separation of the Hamiltonian $\mathcal{H}$ into $\mathcal{H}^{(0)}$ and $h\mathcal{H}^{(1)}$ has formal reasons only, and does not indicate different orders of magnitude. We assume however, that the interactions as $V_{k_1,k_2\,k_3}$ in $\mathcal{H}^{(0)}$ are weak, and that the eigenvalue $E = 0$ of $\mathcal{H}^{(0)}$ is non degenerate, which means that the vacuum state $|0\rangle$ is the only eigenvector with eigenvalue 0 . This can always be achieved by a slight change of some parameters in the higher order interactions of $\mathcal{H}_{in}$.

2.1. Structure of the transformed Hamiltonian

We consider the Hamiltonian (1) and want to transform it by a unitary transformation into a Hamiltonian $\hat{\mathcal{H}}$, so that $\hat{\mathcal{H}}$ is diagonal in the basis of $\mathcal{H}^{(0)}$, or more generally in the case of degeneracy, that

$$[\hat{\mathcal{H}}, \mathcal{H}^{(0)}] = 0. \tag{5}$$

From this commutator and the vacuum state $|0\rangle$ of $\mathcal{H}^{(0)}$,

$$\mathcal{H}^{(0)}|0\rangle = 0, \tag{6}$$

we can draw some conclusions as to the structure of $\hat{\mathcal{H}}$. Combining eqs.(5) and (6) we find

$$\mathcal{H}^{(0)}\hat{\mathcal{H}}|0\rangle = 0, \tag{7}$$

which means that $\hat{\mathcal{H}}|0\rangle$ is an eigenvector of $\mathcal{H}^{(0)}$ with eigenvalue 0. As this eigenvalue is proposed to be non degenerate, the state $\hat{\mathcal{H}}|0\rangle$ must be proportional to $|0\rangle$

$$\hat{\mathcal{H}}|0\rangle = c|0\rangle. \tag{8}$$

Hence the transformed Hamiltonian $\hat{\mathcal{H}}$ written in normal ordering cannot contain products of creation operators only. As it is hermitian, terms with products of annihilation operators only cannot occur either. Claiming that the unitary transformation preserves translation invariance, we finally must have

$$\begin{aligned}\hat{\mathcal{H}} &= E_0 + \hbar \sum_k \hat{\omega}_k\, a_k^\dagger a_k \\ &\quad + \hbar \sum_{k_1\,k_2\,k_3} (\hat{V}_{k_1,k_2\,k_3} a_{k_1}^\dagger a_{k_2} a_{k_3} + \hat{V}^*_{k_1,k_2\,k_3} a_{k_2}^\dagger a_{k_3}^\dagger a_{k_1}) + \cdots,\end{aligned} \tag{9}$$

where the dots indicate interactions of higher order.

2.2. Choice of unitary transformation

For the calculation of the desired unitary transformation we use the standard adiabatic formulation [13], but take such a form that in evaluating $\hat{\mathcal{H}}$ we need not separate singular phase factors. This can be achieved in the following way. Introduce the time dependent Hamiltonian

$$\mathcal{H}(t) = \mathcal{H} - \exp\,(-\epsilon t)\, h\, \mathcal{H}^{(1)} \tag{10}$$

with $\epsilon > 0$ and consider the unitary transformation generated by

$$\frac{\partial}{\partial t}\mathcal{U}(t,t_0) = -i\hbar^{-1}\mathcal{H}(t)\,\mathcal{U}(t,t_0) \tag{11}$$

with

$$\mathcal{U}(t_0,t_0) = 1. \tag{12}$$

Then the Tani equations [13] for the interaction representation,

$$\mathcal{U}_I(t,t_0) = \exp\left(i\hbar^{-1}\mathcal{H}(\infty)\,t\right)\mathcal{U}(t,t_0)\,\exp\left(-i\hbar^{-1}\mathcal{H}(\infty)\,t_0\right), \tag{13}$$

allow to factor $\mathcal{U}_I(0,\infty)$ into a product of two unitary operators

$$\mathcal{U}_I(0,\infty) = \mathcal{T}(\epsilon)\,\mathcal{O}(\epsilon), \tag{14}$$

so that $\mathcal{T}(0)$ exists and transforms the basis of $\mathcal{H}(\infty)$ into the basis of $\mathcal{H}(0)$, whereas $\mathcal{O}(\epsilon)$ contains the singular phase factors and commutes with $\mathcal{H}(\infty)$. For our choice of $\mathcal{H}(\infty) = \mathcal{H}$ and $\mathcal{H}(0) = \mathcal{H}^{(0)}$ it follows from (14) that

$$\mathcal{U}_I(0,\infty)\,\mathcal{H}\,\mathcal{U}_I(0,\infty)^\dagger = \mathcal{T}(\epsilon)\,\mathcal{H}\,\mathcal{T}(\epsilon)^\dagger \tag{15}$$

exists for $\epsilon \to 0_+$ and commutes with $\mathcal{H}^{(0)}$. Hence the transformation of $\mathcal{H}$ into

$$\hat{\mathcal{H}} = \mathcal{T}(0)\,\mathcal{H}\,\mathcal{T}(0)^\dagger = \lim_{\epsilon\to 0_+}\mathcal{U}_I(0,\infty)\,\mathcal{H}\,\mathcal{U}_I(0,\infty)^\dagger \tag{16}$$

has all properties we need. The translation invariance is conserved as follows from the generator (11). The expression for $\hat{\mathcal{H}}$ can further be simplified by replacing the two time limit in eq.(16) by a one time limit and going back to the Schrödinger $\mathcal{U}(0,t)$ of eq.(11). This is possible as it holds

$$\begin{aligned}\mathcal{U}_I(0,t)\,\mathcal{H}\,\mathcal{U}_I(0,t)^\dagger = {} & (\mathcal{U}_I(0,t) - \mathcal{U}_I(0,\infty))\,\mathcal{H}\,(\mathcal{U}_I(0,t)^\dagger - \mathcal{U}_I(0,\infty)^\dagger) \\ & + \mathcal{U}_I(0,t)\,\mathcal{H}\,\mathcal{U}_I(0,\infty)^\dagger + \mathcal{U}_I(0,\infty)\,\mathcal{H}\,(\mathcal{U}_I(0,t)^\dagger \\ & - \mathcal{U}_I(0,\infty)\,\mathcal{H}\,\mathcal{U}_I(0,\infty)^\dagger.\end{aligned} \tag{17}$$

Regarding eq.(17) and $\mathcal{U}_I(0,t)^\dagger = \mathcal{U}_I(t,0)$ together with eq.(13) we therefore may rewrite eq.(16) as

$$\hat{\mathcal{H}} = \lim_{\epsilon\to 0_+}\lim_{t\to\infty}\mathcal{U}(t,0)^\dagger\,\mathcal{H}\,\mathcal{U}(t,0). \tag{18}$$

This is our basic relation, but some comment on degeneracy should be made. The transformation properties of $\mathcal{U}_I(0,\infty)$ and the decomposition (14) usually are proofed for a non degenerate spectrum of $\mathcal{H}(\infty)$ which by some additional interaction always can be reached. In our case, however, we will admit degeneracy from the beginning as to be able to apply the formalism also for $\mathcal{H}_{in} = 0$. It is not difficult to see from the Tani equations that (14) can be generalized to a degenerate spectrum of $\mathcal{H}(\infty)$, so that (18) still exists. It therefore remains to show, that (18) commutes with $\mathcal{H}^{(0)}$. This property can also be deduced from the expansion of (18) into powers of $\mathcal{H}^{(1)}$ for fixed $\mathcal{H}^{(0)}$ and will explicitly be demonstrated in Appendix A. Thus we conclude that (18) and (5) generally hold, as long as both $\mathcal{H}^{(0)}$ and $\mathcal{H}$ have discrete spectra and a perturbation expansion is possible. Then in our case (18) must have the form (9).

2.3. Liouville formulation

For the evaluation of $\hat{\mathcal{H}}$ it is convenient to use a Liouville description. Let us introduce the Liouville operator corresponding to the Hamiltonian by the commutator

$$L \cdots = \hbar^{-1} [\mathcal{H}, \cdots], \tag{19}$$

and in the same way the Liouvillians corresponding to the parts (1) of $\mathcal{H}$ and $\mathcal{H}(t)$. Defining $U(t)$ in Liouville space by

$$U(t) \cdots = \mathcal{U}(t,0)^\dagger \cdots \mathcal{U}(t,0), \tag{20}$$

we may rewrite eq.(18) as

$$\hat{\mathcal{H}} = \lim_{\epsilon \to 0_+} \lim_{t \to \infty} U(t)\, \mathcal{H}, \tag{21}$$

where it follows from eqs.(20) and (11) that

$$\frac{\partial}{\partial t} U(t) = i\, U(t)\, L(t). \tag{22}$$

Using eqs.(19) and (22), the Hamiltonian (21) may be rewritten as

$$\begin{aligned} \hat{\mathcal{H}} &= \mathcal{H}^{(0)} + h\mathcal{H}^{(1)} + \lim_{\epsilon \to 0_+} \int_0^\infty \frac{\partial}{\partial t} U(t)\, (\mathcal{H}^{(0)} + h\mathcal{H}^{(1)}) dt \\ &= \mathcal{H}^{(0)} + h\mathcal{H}^{(1)} - i \lim_{\epsilon \to 0_+} \int_0^\infty \exp\,(-\epsilon t) U(t)\, h\, L^{(1)}\, \mathcal{H}^{(0)} dt \\ &= \mathcal{H}^{(0)} + h\mathcal{H}^{(1)} - i \lim_{\epsilon \to 0_+} W(\epsilon) L^{(1)} \mathcal{H}^{(0)} h \end{aligned} \tag{23}$$

where

$$W(s) = \int_0^\infty \exp(-st) U(t) dt, \tag{24}$$

denotes the Laplace transform of the evolution operator. According to eqs.(22) and (10) it obeys

$$sW(s) - 1 = i \left(W(s)\, (L^{(0)} + h\, L^{(1)}) - W(s+\epsilon)\, h\, L^{(1)} \right). \tag{25}$$

3. Special system - transformation of the Hamiltonian

In this section we will calculate $\hat{\mathcal{H}}$ for the case of the Hamiltonian (1). The key idea consists in restricting the basis in Liouville space to the subspace which is relevant in the case without internal interaction. This means that matrix elements in Liouville space are factored in the way which is exact for the system with $\mathcal{H}_{in} = 0$. For simplicity we restrict to the case that the Hamiltonian is invariant to the transformation $\{a_k \to a_{-k}, a_k^\dagger \to a_{-k}^\dagger\}$, or $\omega_k = \omega_{-k}$ and corresponding relations for the other coefficients. A generalization is straightforward.

3.1. Approximation scheme

For the calculation of $\hat{\mathcal{H}}$ from eq.(23) we must find $-iW(\epsilon)L^{(1)}\mathcal{H}^{(0)}$. Here we rewrite eq.(23) as

$$\hat{\mathcal{H}} = \lim_{\epsilon\to 0}\{\mathcal{H}^{(0)} + i\epsilon(L^{(0)}+i\epsilon)^{-1}\mathcal{H}^{(1)}h - i\{W(\epsilon) - i(L^{(0)}+i\epsilon)^{-1}\}\, L^{(1)}\,\mathcal{H}^{(0)}h\}, \tag{26}$$

where $L^{(1)}\,\mathcal{H}^{(0)} = -L^{(0)}\,\mathcal{H}^{(1)}$ has been employed. Since the last contribution is already of higher order in h we are going to neglect the interaction $\mathcal{H}_{in}$ in $\mathcal{H}^{(0)}$ which just gives rise to a small additive contribution||.

The idea for our approximation scheme results from the Laplace transform $W(s)$ (25) written in the form

$$W(n\epsilon) = i(L^{(0)}+in\epsilon)^{-1} + (W(n\epsilon+\epsilon) - W(n\epsilon))\, L^{(1)}\,(L^{(0)}+in\epsilon)^{-1}\, h. \tag{27}$$

It shows that there are two aspects, the mapping by $L^{(1)}$ and by the resolvents. Consider the space spanned by the basis vectors $a_k^\dagger a_k, a_{-k}^\dagger a_{-k}, a_k^\dagger a_{-k}^\dagger, a_k a_{-k}, 1$. Then the operator $L^{(1)}$ does not lead out of this space, and without internal interaction $\mathcal{H}_{in} = 0$, these basis vectors are eigenvectors of the resolvents. Therefore in this case, the system for $W(m\epsilon)\mathcal{F}_\nu$, $\mathcal{F}_\nu \in \{a_k^\dagger a_k, a_{-k}^\dagger a_{-k}, a_k^\dagger a_{-k}^\dagger, a_k a_{-k}, 1\}$ is closed.

Our approximation now consists in restricting $(L^{(0)}+in\epsilon)^{-1}\mathcal{F}_\nu$ to the contribution proportional to $\mathcal{F}_\nu$ in normal ordering, or in other words, we keep the bare eigenvectors of the resolvents, and replace the eigenvalues by "diagonal elements". In formal terms the approximation means

$$(L^{(0)}+in\epsilon)^{-1}\,\mathcal{F}_\nu = \mathcal{F}_\nu\,(\widetilde{\mathcal{F}_\nu}|(L^{(0)}+in\epsilon)^{-1}\,\mathcal{F}_\nu) + \cdots, \qquad \mathcal{F}_\nu \in \{a_k^\dagger a_k, a_{-k}^\dagger a_{-k}, a_k^\dagger a_{-k}^\dagger, a_k a_{-k}\}, \tag{28}$$

where for expressing the coefficients in such an expansion we have used an inner product for operators $\mathcal{A}$, $\mathcal{B}$

$$(\mathcal{A}|\mathcal{B}) = \mathrm{Tr}(\mathcal{A}^\dagger\mathcal{B}) \tag{29}$$

and suitable dual operators $\widetilde{\mathcal{F}}_\nu$. Using e.g. Glauber states it is quite a simple task to obtain such dual elements of normally order products. In our case it is sufficient to note that

$$\begin{aligned}
\widetilde{a_k^\dagger a_{-k}^\dagger} &= a_k^\dagger a_{-k}^\dagger\,|0\rangle\langle 0| \\
\widetilde{a_k a_{-k}} &= |0\rangle\langle 0|\, a_k a_{-k} \\
\widetilde{a_k^\dagger a_k} &= a_k^\dagger\,|0\rangle\langle 0|\, a_k - |0\rangle\langle 0|.
\end{aligned} \tag{30}$$

For algebraic reasons it is useful to work with linear combinations of the $\mathcal{F}_\nu$. Let us define

$$\begin{aligned}
\mathcal{G}_k^0 &= a_k^\dagger a_k + a_{-k}^\dagger a_{-k} \\
\mathcal{G}_k^+ &= |V_k|^{-1}\,(V_k\, a_k^\dagger a_{-k}^\dagger + V_k^*\, a_k a_{-k}) \\
\mathcal{G}_k^- &= |V_k|^{-1}\,(V_k\, a_k^\dagger a_{-k}^\dagger - V_k^*\, a_k a_{-k})
\end{aligned} \tag{31}$$

|| Such a term finally gives rise to a renormalisation of the many particle interactions in the effective Hamiltonian.

then we have the exact relations

$$
\begin{aligned}
L^{(1)}\,\mathcal{G}_k^0 &= -2|V_k|\,\mathcal{G}_k^- \\
L^{(1)}\,\mathcal{G}_k^+ &= 0 \\
L^{(1)}\,\mathcal{G}_k^- &= 2|V_k|\,(\mathcal{G}_k^0+1),
\end{aligned}
\tag{32}
$$

and the approximations of the resolvents (28) read

$$
\begin{aligned}
(L^{(0)}+in\epsilon)^{-1}\,\mathcal{G}_k^0 &= \mathcal{G}_k^0\,w_k^{00}(n\epsilon) \\
(L^{(0)}+in\epsilon)^{-1}\,\mathcal{G}_k^+ &= \mathcal{G}_k^+\,w_k^{++}(n\epsilon)+\mathcal{G}_k^-\,w_k^{-+}(n\epsilon) \\
(L^{(0)}+in\epsilon)^{-1}\,\mathcal{G}_k^- &= \mathcal{G}_k^+\,w_k^{+-}(n\epsilon)+\mathcal{G}_k^-\,w_k^{--}(n\epsilon),
\end{aligned}
\tag{33}
$$

with

$$
\begin{aligned}
w_k^{00}(n\epsilon) = w_{-k}^{00}(n\epsilon) \quad &= \langle 0|a_k\,\{(L^{(0)}+in\epsilon)^{-1}a_k^\dagger a_k\}a_k^\dagger|0\rangle \\
w_k^{++}(n\epsilon) = w_k^{--}(n\epsilon) = &-in\epsilon\langle 0|a_k a_{-k}\,\{[(L^{(0)})^2+n^2\epsilon^2]^{-1}a_k^\dagger a_{-k}^\dagger\}|0\rangle \\
w_k^{+-}(n\epsilon) = w_k^{-+}(n\epsilon) = &\langle 0|a_k a_{-k}\,\{L^{(0)}[(L^{(0)})^2+n^2\epsilon^2]^{-1}a_k^\dagger a_{-k}^\dagger\}|0\rangle.
\end{aligned}
\tag{34}
$$

The physical meaning of the approximations (33) to (34) can be summarized as follows: The neglected terms vanish with $\mathcal{H}_{in}\to 0$, so the approximation is useful for weak internal interactions. On the other hand due to the normal ordering, the matrix elements $w_k^{\mu\nu}(n\epsilon)$ contain contributions of all orders in $\mathcal{H}_{in}$, which is important for dealing with the resonant parts of the interaction. For the following treatment in the next subsection no further approximations are needed, but it must be shown that the resulting equations are compatible with the limit $\epsilon\to 0$.

3.2. The limit $\epsilon\to 0$

Lead by the form of eq.(26) we introduce the static fluctuation

$$
\begin{aligned}
\mathcal{G}_k^{\nu,n}(\epsilon) = &-i[(W(n\epsilon)-i(L^{(0)}+in\epsilon)^{-1}]L^{(1)}\mathcal{G}_k^\nu h \\
&+i\langle 0|\left\{[(W(n\epsilon)-i(L^{(0)}+in\epsilon)^{-1}]L^{(1)}\mathcal{G}_k^\nu\right\}|0\rangle h, \quad \nu=0,-
\end{aligned}
\tag{35}
$$

and establish a set of equations for this quantities. The Hamiltonian (26) can be written as

$$
\hat{\mathcal{H}} = E_0+\mathcal{H}^{(0)}+1/2\sum_{k,\nu=\pm}\hbar|V_k|\mathcal{G}_k^\nu\lim_{\epsilon\to 0}i\epsilon w_k^{\nu+}(\epsilon)+1/2\sum_k\hbar\omega_k\mathcal{G}_k^{0,1}(0_+)+\cdots \tag{36}
$$

and the set of equations for $\mathcal{G}_k^{\nu,n}(\epsilon)$ is directly obtained from eqs.(27) and (33) yielding

$$
\mathcal{G}_k^{0,n}(\epsilon) \;=\; -2|V_k|w_k^{--}(n\epsilon)\,h\{\mathcal{G}_k^{-,n+1}(\epsilon)-\mathcal{G}_k^{-,n}(\epsilon)\}+\mathcal{I}_k^{0,n}(\epsilon) \tag{37}
$$

$$
\mathcal{G}_k^{-,n}(\epsilon) = 2|V_k|w_k^{00}(n\epsilon)\,h\{\mathcal{G}_k^{0,n+1}(\epsilon)-\mathcal{G}_k^{0,n}(\epsilon)\}+\mathcal{I}_k^{-,n}(\epsilon) \tag{38}
$$

where the inhomogeneous terms are given by

$$
\mathcal{I}_k^{0,n}(\epsilon) = -4|V_k|^2\{w_k^{00}(n\epsilon+\epsilon)-w_k^{00}(n\epsilon)\}w_k^{--}(n\epsilon)\,h^2\mathcal{G}_k^0 \tag{39}
$$

and

$$
\begin{aligned}
\mathcal{I}_k^{-,n}(\epsilon) = &-4|V_k|^2\{w_k^{+-}(n\epsilon+\epsilon)-w_k^{+-}(n\epsilon)\}w_k^{00}(n\epsilon)\,h^2\mathcal{G}_k^+ \\
&-4|V_k|^2\{w_k^{--}(n\epsilon+\epsilon)-w_k^{--}(n\epsilon)\}w_k^{00}(n\epsilon)\,h^2\mathcal{G}_k^-.
\end{aligned}
\tag{40}
$$

Solving for $\mathcal{G}_k^{0,n}(\epsilon)$ we eliminate $\mathcal{G}_k^{-,n}(\epsilon)$ to obtain

$$\begin{aligned}\mathcal{G}_k^{0,n}(\epsilon) = & -4|V_k|^2 w_k^{--}(n\epsilon) w_k^{00}(n\epsilon+\epsilon) h^2 \{\mathcal{G}_k^{0,n+2}(\epsilon) - \mathcal{G}_k^{0,n+1}(\epsilon)\} \\ & + 4|V_k|^2 w_k^{--}(n\epsilon) w_k^{00}(n\epsilon) h^2 \{\mathcal{G}_k^{0,n+1}(\epsilon) - \mathcal{G}_k^{0,n}(\epsilon)\} \\ & + \mathcal{I}_k^{0,n}(\epsilon) - 2|V_k| w_k^{--}(n\epsilon)\{\mathcal{I}_k^{-,n+1}(\epsilon) - \mathcal{I}_k^{-,n}(\epsilon)\} h. \end{aligned} \tag{41}$$

In this equation the limits $\epsilon \to 0$ can be treated. One uses the fact that in the matrix elements $w_k^{--}(n\epsilon)$ (cf. eq.(34)) there is an explicit factor ϵ, and $L^{(0)}$ produces energy differences with respect to the vacuum state only which have been assumed to be different from zero. Therefore in eq.(41) the products $w_k^{--}(n\epsilon) w_k^{00}(n\epsilon+\epsilon)$ are finite, although $w_k^{00}(n\epsilon)$ diverges with $\epsilon \to 0$.

$$\begin{aligned}\lim_{\epsilon\to 0} w_k^{00}(n\epsilon) w_k^{--}(m\epsilon) = & -m/n \lim_{\epsilon \to 0} \langle 0| a_k\, in\epsilon \,\{(L^{(0)} + in\epsilon)^{-1} a_k^\dagger a_k\} a_k^\dagger |0\rangle \\ & \times \langle 0| a_k a_{-k} \,[(L^{(0)})^2 + m^2\epsilon^2]^{-1}\, a_k^\dagger a_{-k}^\dagger |0\rangle \\ = & -m/n \langle 0| a_k \{P a_k^\dagger a_k\} a_k^\dagger |0\rangle \langle 0| a_k a_{-k} \{(L^{(0)})^{-2} a_k^\dagger a_{-k}^\dagger\} |0\rangle \end{aligned} \tag{42}$$

where P is the projector onto the null space of $L^{(0)}$

$$P = \lim_{\epsilon\to 0} i\epsilon\, (L^{(0)} + i\epsilon)^{-1}. \tag{43}$$

As a consequence, with the abbreviation

$$\hat{h}_k^2 = 4|V_k|^2 \langle 0| a_k \,\{P a_k^\dagger a_k\} a_k^\dagger |0\rangle \langle 0| a_k a_{-k} \{(L^{(0)})^{-2} a_k^\dagger a_{-k}^\dagger\}|0\rangle h^2 \tag{44}$$

eq.(41) for $\epsilon \to 0$ yields

$$\begin{aligned}\mathcal{G}_k^{0,n}(0_+) = & \frac{n}{n+1} \hat{h}_k^2 \{\mathcal{G}_k^{0,n+2}(0_+) - \mathcal{G}_k^{0,n+1}(0_+)\} \\ & - \hat{h}_k^2 \{\mathcal{G}_k^{0,n+1}(0_+) - \mathcal{G}_k^{0,n}(0_+)\} - \hat{h}_k^2 \frac{1}{n+1} \mathcal{G}_k^0. \end{aligned} \tag{45}$$

The solution can be written as

$$\mathcal{G}_k^{0,n}(0_+) = a(n, \hat{h}_k) \mathcal{G}_k^0 \tag{46}$$

with

$$a(n, \hat{h}_k) = -\sum_{m=1}^{\infty} \frac{n!}{(n+2m-1)!} \frac{1}{2m-1} \left(\frac{(2m)!}{m!}\right)^2 4^{-m} \hat{h}_k^{2m}. \tag{47}$$

For $a(n=1, \hat{h}_k)$ the power series can be summed to yield

$$a(1, \hat{h}_k) = \sqrt{1 - \hat{h}_k^2} - 1, \tag{48}$$

so that eq.(36) finally reads

$$\hat{\mathcal{H}} = E_0 + \mathcal{H}^{(0)} + 1/2 \sum_k \hbar\omega_k (\sqrt{1 - \hat{h}_k^2} - 1) \mathcal{G}_k^0. \tag{49}$$

The third term in eq.(36) vanishes. The appearing parameters are determined by the spectrum of $L^{(0)}$ and its eigenvectors. In the case of $\mathcal{H}_{in} = 0$ one observes that

$$P a_k^\dagger a_k = a_k^\dagger a_k \tag{50}$$

and

$$L^{(0)}\, a_k^\dagger a_{-k}^\dagger = 2\omega_k a_k^\dagger a_{-k}^\dagger, \tag{51}$$

so that eq.(49) correctly reduces to the well known result of the Bogoliubov transformation

$$\hat{\mathcal{H}} = E_0 + \sum_k \sqrt{\omega_k^2 - |V_k|^2 h^2}\, a_k^\dagger a_k, \quad (\hat{\mathcal{H}}_{in} \equiv 0). \tag{52}$$

For $\mathcal{H}_{in} \neq 0$ the eigenvalues $2\omega_k$ of $L^{(0)}$ split and the width of this splitting is contained in the threshold parameter $\hat{h}_k^2$.

3.3. Threshold parameter $\hat{h}_k$ in the limit $\Delta k \to 0$

Having found the transformed Hamiltonian (49) we will evaluate the arising parameters $\hat{h}_k$ for a small spacing of the wave vector components Δk by taking their asymptotic values $\Delta k \to 0$ (thermodynamic limit). For $\Delta k \to 0$ the energy spectrum of $\mathcal{H}^{(0)}$ will become dense, so we express our parameters by time dependent correlation functions which then can be taken in their Markovian form. In the definition (44) of $\hat{h}_k$, the expression of interest is given by $\langle 0|a_k\,\{Pa_k^\dagger a_k\}a_k^\dagger|0\rangle\langle 0|a_k a_{-k}\{L^{(0)\,-2}a_k^\dagger a_{-k}^\dagger\}|0\rangle$. For decreasing Δk the second factor of the product will diverge, as eigenvalues of $\mathcal{H}^{(0)}$ will approach zero, while the first factor is expected to tend to zero. We therefore treat the product as a whole.

Using Laplace transform, eq.(44) can be written as

$$\begin{aligned}\hat{h}_k^2 = |V_k|^2 h^2 \lim_{\epsilon\to 0}\Bigg(&\int_0^\infty \exp(-\epsilon t)\langle 0|a_k\{a_k^\dagger(t)a_k(t) + a_k^\dagger(-t)a_k(-t)\}a_k^\dagger|0\rangle dt \\ &\times \int_0^\infty \exp(-\epsilon t)\langle 0|a_k a_{-k}\{a_k^\dagger(t)a_{-k}^\dagger(t) + a_k^\dagger(-t)a_{-k}^\dagger(-t)\}|0\rangle dt\Bigg),\end{aligned} \tag{53}$$

where

$$a_k(t) = \exp(iL^{(0)}t)a_k = \exp(i\hbar^{-1}\,\mathcal{H}^{(0)}t)\,a_k\,\exp(-i\hbar^{-1}\,\mathcal{H}^{(0)}t) \tag{54}$$

denotes the Heisenberg dynamics. The correct order of the limits, i.e. first $\epsilon \to 0$ and then $\Delta k \to 0$ is in general important for our considerations. Employing properties of Laplace transforms (cf. Appendix B) we can however show that we can neglect here the factors $\exp(-\epsilon t)$ provided that in the thermodynamic limit the time dependent correlation functions in the integrands decay to zero for $t \to \infty$ sufficiently fast. Thus the result essentially corresponds to a change of the order of the limits.

The correlation functions in (53) will be calculated by expanding the Heisenberg operators $a_k^\dagger(t)$ into powers of the strength of the interaction $\mathcal{H}_{in}$ for fixed one particle correlation functions $\Xi_k(t)$ [14]

$$a_k^\dagger(t) = a_k^\dagger \Xi_k(t) + \cdots, \tag{55}$$

where

$$\Xi_k(t) = \langle 0|\left\{a_k \exp(iL^{(0)}t)a_k^\dagger\right\}|0\rangle = \langle 0|a_k \exp(i/\hbar\mathcal{H}^{(0)}t)a_k^\dagger|0\rangle. \tag{56}$$

Inserting the lowest order of (55) into (53) which corresponds to a factoring according Wick's theorem, and using the Markovian approximation for $\Xi_k(t)$,

$$\Xi_k(t) = \exp\left(i\omega_k t + i\Delta\omega_k t - \Gamma_k t\right), \tag{57}$$

one finally obtains from (53)

$$\begin{aligned}\hat{h}_k^2 &= 4|V_k|^2 h^2 \int_0^\infty |\Xi_k(t)|^2 dt \times \mathrm{Re} \int_0^\infty \Xi_k(t)\Xi_{-k}(t) dt \\ &= |V_k|^2 \frac{h^2}{(\omega_k + \Delta\omega_k)^2 + \Gamma_k^2}.\end{aligned} \tag{58}$$

This result shows that the values of $\hat{h}_k^2$ are lowered by the damping rates. The explicit damping rates and frequency shifts may be derived from Mori's theory [15] or other methods of many body theory yielding in lowest order

$$\Gamma_{k_1} = \sum_{k_2 k_3} 2|V_{k_1,k_2\,k_3}|^2 \pi\delta(\omega_{k_1} - \omega_{k_2} - \omega_{k_3}) \tag{59}$$

$$\Delta\omega_{k_1} = \sum_{k_2 k_3} 2|V_{k_1,k_2\,k_3}|^2 \,\mathrm{Pr}\, \frac{1}{\omega_{k_1} - \omega_{k_2} - \omega_{k_3}}. \tag{60}$$

In some regions of wave numbers k it can occur that the simple form (59) for Γ_k vanishes, as energy and momentum cannot simultaneously be conserved. In such cases higher order interactions of $\mathcal{H}_{in}$ must be considered in Γ_k, but this does not affect the validity of eq.(58).

4. Result and discussion

We have performed a unitary transformation of the given Hamiltonian $\mathcal{H}$ into a Hamiltonian $\hat{\mathcal{H}}$ which according to eqs.(49) and (31) can be written as

$$\begin{aligned}\hat{\mathcal{H}} &= E_0 + \hbar \sum_k \hat{\omega}_k(h)\, a_k^\dagger a_k \\ &\quad + \hbar \sum_{k_1\,k_2\,k_3} (\hat{V}_{k_1,k_2\,k_3}(h) a_{k_1}^\dagger a_{k_2} a_{k_3} + \hat{V}_{k_1,k_2\,k_3}(h)^* a_{k_2}^\dagger a_{k_3}^\dagger a_{k_1}) + \cdots.\end{aligned} \tag{61}$$

The frequencies $\hat{\omega}_k(h)$ follow from the preceding section and will be discussed later. The higher order interactions $\hat{V}_{k_1,k_2\,k_3}(h)$ have not been considered explicitly. Their origin and oder of magnitude will be considered at the end.

We first will discuss the general result (61) from a point of view which makes the connection to the usual Bogoliubov transformation more transparent. According to eq.(16) we have $\hat{\mathcal{H}} = \mathcal{T}(0)\mathcal{H}\mathcal{T}(0)^\dagger$. Therefore introducing a unitary transformation of the creation, or annihilation operators respectively by

$$\hat{a}_k^\dagger = \mathcal{T}(0)^\dagger a_k^\dagger \mathcal{T}(0) \tag{62}$$

we can perform $\mathcal{T}(0)^\dagger \hat{\mathcal{H}} \mathcal{T}(0)$ and express the original Hamiltonian in terms of operators (62). From eqs.(1) and (61) we explicitly find

$$\hbar \sum_k \omega_k\, a_k^\dagger a_k + h\,\hbar/2(\sum_k V_k\, a_k^\dagger a_{-k}^\dagger + V_k^*\, a_k a_{-k})$$

$$
\begin{aligned}
&+\hbar \sum_{k_1\,k_2\,k_3} (V_{k_1,k_2\,k_3} a^\dagger_{k_1} a_{k_2} a_{k_3} + V^*_{k_1,k_2\,k_3} a^\dagger_{k_2} a^\dagger_{k_3} a_{k_1}) + \cdots \\
&= E_0 + \hbar \sum_k \hat{\omega}_k(h)\, \hat{a}^\dagger_k \hat{a}_k \\
&+\hbar \sum_{k_1\,k_2\,k_3} (\hat{V}_{k_1,k_2\,k_3}(h) \hat{a}^\dagger_{k_1} \hat{a}_{k_2} \hat{a}_{k_3} + \hat{V}_{k_1,k_2\,k_3}(h)^* \hat{a}^\dagger_{k_2} \hat{a}^\dagger_{k_3} \hat{a}_{k_1}) + \cdots . \qquad (63)
\end{aligned}
$$

Inspection of eq.(63) shows that the passage to the new creation and annihilation operators removes the interaction $h\mathcal{H}^{(1)}$ in the Hamiltonian, and gives rise to frequencies and interactions which depend on the coupling strength h. Thus the used transformation has the feature of a Bogoliubov transformation, although we did not try to evaluate (62) as a function of the given $a^\dagger_k$ and a_k . To find this function for non vanishing $\mathcal{H}_{in}$ seems to be extremely difficult, as one is forced to separate the phase factors in the starting time dependent transformation and must find the decomposition (14). Therefore it is a first step to obtain the resulting frequencies $\hat{\omega}_k(h)$. They are sufficient to determine, how the bare threshold for h is modified by the internal interaction $\mathcal{H}_{in}$, and allow for a perturbative treatment of the eigenvalues.

The results of $\hat{\omega}_k(h)$ follow from eqs.(49), (31), (58), to be

$$
\hat{\omega}_k(h) = \sqrt{\frac{(\omega_k + \Delta\omega_k)^2 + \Gamma_k^2 - |V_k|^2 h^2}{(\omega_k + \Delta\omega_k)^2 + \Gamma_k^2}}\, \omega_k . \qquad (64)
$$

The quantities $\Delta\omega_k$ and Γ_k denote the frequency shifts and damping rates of the correlation functions (56).

The essential point of the results for $\hat{\omega}_k(h)$ is, that they show a threshold for the coupling strength, $h^2 \le h^2_{th}$, with

$$
h^2_{th} = \min_k \{ \frac{(\omega_k + \Delta\omega_k)^2 + \Gamma_k^2}{|V_k|^2} \}. \qquad (65)
$$

This value is in accordance with dynamical considerations [7] which have been mentioned in the introduction. The question, what will happen beyond this threshold, cannot generally be answered. In the case of $\mathcal{H}_{in} = 0$, the energy spectrum of $\mathcal{H}$ becomes continuous, and discrete excitations $\hat{\omega}_k(h)$ no longer exist. For interactions $\mathcal{H}_{in} \neq 0$, the behavior will depend on the structure and strength of the higher particle interactions which up to now can just enter into the results via frequency shifts and damping rates. These interactions can preserve the discrete spectrum of $\mathcal{H}_{in}$, which has been assumed from the beginning, but in this case the states for $h > h_{th}$, must be entirely different from those for $\mathcal{H}_{in} = 0$, so that the subspace of the Liouville space used in section 3 must be enlarged. One might speculate, that the situation is similar to that of a phase transition.

We have confined the explicit evaluation in section 3 to the lowest nontrivial order. The evaluation of the interactions $\hat{V}_{k_1,k_2\,k_3}(h)$ can be performed in a similar fashion, e.g. by expanding eq.(26) in powers of creation an annihilation operators and solving the corresponding linear equations in terms of correlation functions. For the subthreshold behaviour $h^2 < h^2_{th}$ these interactions follow from the neglected terms in eqs.(26) and

(33) to be of the order of magnitude of the internal coupling $\mathcal{H}_{in}$ and thus will not play an essential role for the low temperature regime. As just mentioned the interactions become crucial above this threshold, as the spectral structure of the Hamiltonian changes considerably. Since the lowest nontrivial order of our perturbation expansion already breaks down we suppose that such a regime calls for a completely different type of perturbation scheme. We leave this point for further investigation.

5. Outlook

The steady state of periodically driven systems beyond linear response is important for nonlinear resonance experiments in magnetic samples . For the parallel pump in ferromagnetic materials one can use the coupled oscillators treated in this paper [7]. In this case the Hamiltonian consists of the time independent part $\mathcal{H} = \mathcal{H}_{harm} + \mathcal{H}_{in}$ given by eqs.(2) and (3), while the pump term is a generalization of $h\mathcal{H}^{(1)}$

$$\mathcal{H}_p = \hbar\, h/2 \sum_k (\exp(-i\omega t) V_k a_k^\dagger a_{-k}^\dagger + \exp(i\omega t) V_k^* a_k a_{-k}). \tag{66}$$

The formulation of a steady state for this periodically driven system is closely related to the unitary transformation discussed in this paper.

In [9] it was generally shown that for a macroscopic periodic Hamiltonian $\mathcal{H} + h\sum_m \exp(im\omega t)\mathcal{H}_m$, one can describe the time dependency of the statistical operator for times longer than internal relaxation times by a statistical operator which is constructed with help of an adiabatically changing amplitude $\exp(\epsilon t)h$, taking ϵ smaller than the inverse recurrence time (cf. [16, 17, 18] for applications). Omitting details one obtains for the the steady state

$$\rho_{st}(t) = Z^{-1} \exp\{-\beta \mathcal{Q}(t,h)\mathcal{H}_a(\exp(\epsilon t)h)\mathcal{Q}^\dagger(t,h)\}, \tag{67}$$

where the unitary operator $\mathcal{Q}(t,h)$ is nearly periodic with the driving period

$$\mathcal{Q}(t+\tau,h) = \mathcal{Q}(t,\exp(\epsilon\tau)h) \tag{68}$$

and $\mathcal{H}_a(h)$ is to be calculated from

$$\mathcal{H}_a(h) = \lim_{t\to-\infty} \mathcal{U}(t,0)^\dagger \mathcal{H}\mathcal{U}(t,0) \tag{69}$$

with the time evolution

$$\frac{\partial}{\partial t}\mathcal{U}(t,0) = -i\hbar^{-1}\{\mathcal{H} + \exp(\epsilon t)h\sum_m \exp(im\omega t)\mathcal{H}_m\}\mathcal{U}(t,0). \tag{70}$$

The initial state at $t = 0$ just determines the inverse temperature in eq.(67). The result for eq.(69) is similar to the starting formula (18) in section 2. The difference is that in the driven case $\mathcal{H}$ is independent of h , and the equation of motion (70) for $\mathcal{U}(t,0)$ has a further periodic time dependency. These points, however, also allow for the method of section 3, if a suitable subset in Liouville space can be found. In some respect the calculation of (69) is simpler than that of $\hat{\mathcal{H}}$, as the series for $\mathcal{H}_a(h)$ in powers of h is of the type (A.3) . It is also possible to find the power series expansion of the full exponent of $\rho_{st}(t)$ in which only powers of $y = -\exp(\epsilon t)h$ appear

$$\mathcal{H}_{st}(t) = Q(t,h)\mathcal{H}_a(\exp(\epsilon t)h)Q^\dagger(t,h) = \mathcal{H} + y\sum_{m_1}\exp(im_1\omega t)\{L + m_1\omega - i\epsilon\}^{-1}L_{m_1}\mathcal{H}$$
$$+y^2\sum_{m_1,m_2}\exp(i(m_1+m_2)\omega t)\{L + (m_1+m_2)\omega - 2i\epsilon\}^{-1}L_{m_2}\{L + m_1\omega - i\epsilon\}^{-1}L_{m_1}\mathcal{H}$$
$$+\cdots \tag{71}$$

The consequence for our driven system is that, neglecting $\mathcal{H}_{in}$ in eq.(71), one has the same subspace as used section 3. Therefore for $\mathcal{H} = \mathcal{H}_{harm}$ with the pump term (66), one can exactly evaluate the exponent (71) of the statistical operator which then reads

$$\mathcal{H}_{st}(t)|_{\epsilon\to 0} = \text{const.} + \sum_k \frac{\hbar\omega_k \text{sign}(\omega_k - \omega/2)}{\sqrt{(\omega_k - \omega/2)^2 - |V_k|^2h^2}}$$
$$\times\{(\omega_k - \omega/2)a_k^\dagger a_k + h/2(\exp(-i\omega t)V_k a_k^\dagger a_{-k}^\dagger + \exp(i\omega t)V_k^* a_k a_{-k})\}, \tag{72}$$

provided $\min_k\{|V_k|^{-2}(\omega_k - \omega/2)^2\} \neq 0$. For the case $\mathcal{H}_{in} \neq 0$, one expects, that an approximation corresponding to eq.(28), will modify the result (72) by damping rates and frequency shifts similar to eq.(64). For this calculation however, the scaling (42) with an ϵ smaller than the inverse recurrence time together with the thermodynamic limit must be reexamined.

A former attempt to calculate ρ_{st} for the parallel pump [19] was restricted to the case of a time independent rotated Hamiltonian. Regarding four magnon interactions, the author could show, that this interaction stabilizes the system in a way, which is known from the very successful phenomenological S–Theory [7]. The damping rates, however, could not be incorporated in ρ_{st} , so that the threshold for the driving amplitude was reduced to zero. As the approach of our paper provides a possibility to take the damping rates in the Hamiltonian into account, one can expect that taking all these ideas together, there is a chance to find a complete microscopic formulation for the steady state of a parallel pumped system.

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Appendix A. Properties of the transformed Hamiltonian

It will be shown that the transformed Hamiltonian $\hat{\mathcal{H}}$ commutes with $\mathcal{H}^{(0)}$. Further it is pointed out that in some cases $\hat{\mathcal{H}}$ can be obtained from a set of differential equations with respect to the coupling.

Appendix A.1. Differential equation for $U(\infty)\mathcal{H}$

According to eq.(21) we must find $\lim_{t\to\infty} U(t)\mathcal{H}$. Therefore a differential equation for this quantity is established. For the formalism of section 2 the special decomposition (1) of $\mathcal{H}$ is not needed. Thus for fixed h in $\mathcal{H}$ we introduce

$$L_x(t) = L - x\exp(-\epsilon t)L^{(1)}, \tag{A.1}$$

and make use of $L_{x=h}(t) = L(t)$ and $L_{x=0}(t) = L$ at the end. The equation of motion for $U_x(t)$ is defined as

$$\frac{\partial}{\partial t}U_x(t) = iU_x(t)\,L_x(t). \tag{A.2}$$

So from its Laplace transform one easily finds the expansion of $\lim_{t\to\infty} U_x(t)\mathcal{H} = \lim_{s\to 0} sW_x(s)\mathcal{H}$ into powers of x

$$\begin{aligned}\lim_{t\to\infty} U(t)\mathcal{H} \quad &= \mathcal{H} + x\,(L+i\epsilon)^{-1}L^{(1)}\,\mathcal{H} + x^2\,(L+2i\epsilon)^{-1}L^{(1)}\,(L+i\epsilon)^{-1}L^{(1)}\,\mathcal{H}\\ &\quad + x^3\,(L+3i\epsilon)^{-1}L^{(1)}\,(L+2i\epsilon)^{-1}L^{(1)}\,(L+i\epsilon)^{-1}L^{(1)}\,\mathcal{H} + \cdots.\end{aligned} \tag{A.3}$$

Applying $L - x\,L^{(1)}$ to this series yields

$$i\,(L - x\,L^{(1)})\,\lim_{t\to\infty} U_x(t)\mathcal{H} = \epsilon x\frac{\partial}{\partial x}\lim_{t\to\infty} U_x(t)\mathcal{H}. \tag{A.4}$$

This equation can also be obtained without explicitly using the power series. For the validity of the result it is important that the limit $t\to\infty$ and the derivative with respect to x can be interchanged.

Appendix A.2. Consequences for $\hat{\mathcal{H}}$

Let us define

$$\hat{\mathcal{H}}_x = \lim_{\epsilon\to 0}\lim_{t\to\infty} U_x(t)\,\mathcal{H}, \tag{A.5}$$

then one sees that

$$\hat{\mathcal{H}}_{x=h} = \hat{\mathcal{H}}, \qquad \hat{\mathcal{H}}_{x=0} = \mathcal{H}. \tag{A.6}$$

Taking $\epsilon\to 0$ in (A.4) implies

$$(L - xL^{(1)})\,\hat{\mathcal{H}}_x = 0. \tag{A.7}$$

This means, that for $x = h$ it must hold

$$[\mathcal{H}^{(0)}, \hat{\mathcal{H}}] = 0. \tag{A.8}$$

The differential equation (A.4) does not only imply the commutator of $\hat{\mathcal{H}}$ with $\mathcal{H}^{(0)}$, but can also serve, to determine $\hat{\mathcal{H}}$ itself. Define the projection operator P_x onto the null space of $L - xL^{(1)}$, then it follows from (A.7) that

$$P_x\hat{\mathcal{H}}_x = \hat{\mathcal{H}}_x \tag{A.9}$$

whereas eqs.(A.4) and (A.5) yield

$$P_x\frac{\partial}{\partial x}\hat{\mathcal{H}}_x = 0. \tag{A.10}$$

If one has a basis of the null space of $L - xL^{(1)}$, eq.(A.10) provides a set of differential equations for the coefficients of the basis vectors, and allows to calculate $\hat{\mathcal{H}}_{x=h} = \hat{\mathcal{H}}$. We will illustrate this possibility for two simple examples.

The first example again shows the general form of $\hat{\mathcal{H}}$, although the method is not appropriate for practical use. Let have $\mathcal{H}_x = \mathcal{H} - x\mathcal{H}^{(1)} = \mathcal{H}^{(0)} + (h-x)\mathcal{H}^{(1)}$ an energy spectrum which is not degenerate

$$\mathcal{H}_x|n_x\rangle = E_n(h-x)|n_x\rangle, \quad \langle n_x|m_x\rangle = \delta_{nm}, \tag{A.11}$$

then P_x acting onto $\mathcal{F}$ in Hilbert space gives $P_x\mathcal{F} = \sum_m |m_x\rangle\langle m_x|\mathcal{F}|m_x\rangle\langle m_x|$, so that from eqs.(A.9) and (A.10) it follows

$$\hat{\mathcal{H}}_x = \sum_n c_n(x)|n_x\rangle\langle n_x|, \qquad P_x\frac{\partial}{\partial x}\hat{\mathcal{H}}_x = \sum_n c_n'(x)|n_x\rangle\langle n_x| = 0, \tag{A.12}$$

i.e. $c_n'(x) = 0$. Together with $c_n(x) = c_n(0) = E_n(h)$ it therefore results

$$\hat{\mathcal{H}} = \hat{\mathcal{H}}_{x=h} = \sum_n E_n(h)|n^{(0)}\rangle\langle n^{(0)}| \tag{A.13}$$

which means that the Hamiltonian $\mathcal{H}$ is transformed to the eigenbasis $\{|n_{x=h}\rangle = |n^{(0)}\rangle\}$ of $\mathcal{H}^{(0)}$. It is clear that this is just another way to obtain the result of the adiabatic theorem. In the approach through eq.(A.10) however, diverging phase factors are eliminated from the beginning.

If the null space of $L - xL^{(1)}$ occurring in eq.(A.10) can be restricted to a finite subspace, then the resulting differential equations may be useful to calculate $\hat{\mathcal{H}}$. We will illustrate this point dealing with an example. For simplicity we take one oscillator with

$$\mathcal{H}^{(0)} = \hbar\omega\, a^\dagger a, \quad \mathcal{H}^{(1)} = \hbar/2\, V(a^\dagger a^\dagger + aa). \tag{A.14}$$

As $L - xL^{(1)}$ does not lead out of the space spanned by $a^\dagger a^\dagger, aa, a^\dagger a, 1$, P_x acting onto this space produces a linear combinations of the two possible operators, $\mathcal{H}_x = \mathcal{H}^{(0)} + (h-x)\mathcal{H}^{(1)}$, and unity. We therefore can write

$$\hat{\mathcal{H}}_x = c_0(x) + c(x)\mathcal{H}_x \tag{A.15}$$

with the initial condition $\hat{\mathcal{H}}_{x=0} = \mathcal{H}_{x=0}$. This ansatz fulfills eq.(A.9), whereas eq.(A.10) yields

$$P_x\frac{\partial}{\partial x}\hat{\mathcal{H}}_x = c_0'(x) + c'(x)\,\mathcal{H}_x - c(x)P_x\mathcal{H}^{(1)} = 0. \tag{A.16}$$

So with the explicit expression for $P_x\mathcal{H}^{(1)}$

$$P_x\mathcal{H}^{(1)} = \lim_{\epsilon\to 0} i\epsilon(L - xL^{(1)} + i\epsilon)^{-1}\mathcal{H}^{(1)} = -V^2(h-x)\{\omega^2 - V^2(h-x)^2\}^{-1}(\hbar\omega/2 + \mathcal{H}_x) \tag{A.17}$$

one obtains the differential equations

$$\begin{aligned} 0 &= c'(x) + V^2(h-x)\{\omega^2 - V^2(h-x)^2\}^{-1}c(x) \\ 0 &= c_0'(x) + \hbar\omega V^2(h-x)\{2(\omega^2 - V^2(h-x)^2)\}^{-1}c(x). \end{aligned} \tag{A.18}$$

Integration with the initial conditions $c(0) = 1$ and $c_0(0) = 0$ yield the well known result

$$\begin{aligned}\hat{\mathcal{H}} &= \hat{\mathcal{H}}_{x=h} = c_0(h) + c(h)\mathcal{H}^{(0)} \\ &= \hbar/2\,(\sqrt{\omega^2 - V^2h^2} - \omega) + \hbar\,\sqrt{\omega^2 - V^2h^2}a^\dagger a.\end{aligned} \tag{A.19}$$

It remains the question, whether the differential equations (A.10) can be used in more general cases by taking an ansatz in a restricted null space, $\hat{\mathcal{H}}_x = \sum'_\nu c_\nu(x) P_x \mathcal{G}_\nu$, and truncating $P_x \frac{\partial}{\partial x} \hat{\mathcal{H}}_x$. There seems to be chance to find such an approach for the system treated in the text, as our basic results follow from algebraic methods for inverting Liouville operators.

Appendix B. On the thermodynamic limit

Let

$$\begin{aligned}f_{\Delta k}(t) &= \sum_\Omega f_\Omega \exp(i\Omega t) \\ g_{\Delta k}(t) &= \sum_{\Omega'} g_{\Omega'} \exp(i\Omega' t)\end{aligned} \tag{B.1}$$

denote the correlation functions appearing in eq.(53) for a system of finite size. We assume that sensible thermodynamic limits $\Delta k \to 0$ exist. Time reversal symmetry of the correlation functions and the properties of the vacuum state imply that

$$f_\Omega = f_{-\Omega}, \quad g_{\Omega'} = g_{-\Omega'}, \quad g_{\Omega'=0} = 0. \tag{B.2}$$

The spectrum of the convolution which is determined by

$$f_{\Delta k}(t) \otimes g_{\Delta k}(t) = \int_{-\infty}^{\infty} F_{\Delta k}(\omega) \exp(i\omega t)\, d\omega \tag{B.3}$$

consists of course of a sum of distributions

$$F_{\Delta k}(\omega) = \sum_{\Omega \neq \Omega'} \frac{\delta(\omega - \Omega) - \delta(\omega - \Omega')}{i(\Omega - \Omega')} f_\Omega g_{\Omega'} + i \sum_{\Omega' \neq 0} \delta'(\omega - \Omega') f_{\Omega'} g_{\Omega'}. \tag{B.4}$$

Applying Laplace transform to eq.(B.3) and taking the limit $s \to 0$ we arrive at

$$\begin{aligned}&\lim_{s\to 0} \left(\int_0^\infty \exp(-st) f_{\Delta k}(t)\, dt \int_0^\infty \exp(-st) g_{\Delta k}(t)\, dt \right) \\ &= \lim_{s\to 0} \int_{-\infty}^{\infty} \frac{1}{s - i\omega} F_{\Delta k}(\omega)\, d\omega = -\int_{-\infty}^{\infty} \frac{1}{i\omega} F_{\Delta k}(\omega)\, d\omega.\end{aligned} \tag{B.5}$$

For the last step we need that the final integral exists. Straightforward algebra using the properties (B.2) shows that the value of the integral is given by $f_{\Omega=0} \sum_{\Omega' \neq 0} g_{\Omega'}/(\Omega')^2$ and that it remains finite in the thermodynamic limit. If we now perform the thermodynamic limit $\Delta k \to 0$ in eq.(B.5) we obtain

$$\begin{aligned}&\lim_{\Delta k \to 0} \lim_{s\to 0} \left(\int_0^\infty \exp(-st) f_{\Delta k}(t)\, dt \int_0^\infty \exp(-st) g_{\Delta k}(t)\, dt \right) \\ &= -\int_{-\infty}^{\infty} \frac{1}{i\omega} F_{\Delta k \to 0}(\omega)\, d\omega = \int_0^\infty f_{\Delta k \to 0}(t)\, dt \int_0^\infty g_{\Delta k \to 0}(t)\, dt.\end{aligned} \tag{B.6}$$

The last expression just follows using the inverse Fourier transform of eq.(B.3) and observing that the spectrum according to eqs.(B.2) and (B.4) obeys the symmetry $F_{\Delta k \to 0}(\omega) = -F_{\Delta k \to 0}(-\omega)$.

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