# Analytical investigation of modulated time–delayed feedback control

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**Abstract.** The influence of time–dependent control parameters on time–delayed feedback schemes for control of chaos is investigated by analytical means. For the logistic map the linear stability of the period–two orbit subjected to a modulated time–delayed feedback loop is calculated. We find enhanced control performance due to phase lags between the periodic orbit and the controller.

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## 1. Introduction

Although control of chaos by time–delayed feedback has been introduced more than one decade ago [1] it is still one of the most active fields in applied nonlinear science [2]. Several variants of the original time–delayed feedback scheme have been proposed to improve the control performance. Among those are schemes employing multiple delays to stabilise strongly unstable orbits [3] or schemes using modulation of the control parameters with a period different from the periodic target state [4, 5] to overcome the so called odd number limitation from which time–delayed feedback control suffers [6]. It has been reported recently that time–dependent modulation improves the control performance in autonomous systems by several orders of magnitude [7, 8]. Whereas in autonomous systems the phase of a periodic target state does not play a significant role a phase selection mechanism may take place when a time–dependent control loop is applied. Thus removing the Goldstone mode of the original dynamics the control performance may be enhanced. We study the very basic nature of this mechanism with analytic tools by investigating the simplest time discrete model subjected to time–delayed feedback control. Applying standard linear stability analysis we discuss in detail the change in control performance induced by the external time–dependence of the control loop.

## 2. Modulated control of the logistic map

Consider the logistic map subjected to time–delayed feedback control for stabilising a periodic orbit of period $p$. The equation of motion reads

$$x_{n+1} = 1 - \mu x_n^2 + K_n(x_n - x_{n-p}) \quad . \tag{1}$$

Whereas the original Pyragas scheme corresponds to a fixed value of the control amplitude $K_n \equiv K$ we are here considering the influence of a time–dependent parameter having the same period as the target state, $K_n = K_{n+p}$. We are investigating the simplest nontrivial case‡ $p = 2$, i.e. control of the period–two orbit $x^*_{0/1} = (1 \pm \sqrt{4\mu - 3})/(2\mu)$. The periodic modulation of the control amplitude reads $K_{2\ell} = K_0$, $K_{2\ell+1} = K_1$. It is already obvious that for such a choice the phase of the orbit matters. Whereas for the plain logistic map the "two" orbits $x^*_0, x^*_1, x^*_0, x^*_1, \ldots$ and $x^*_1, x^*_0, x^*_1, x^*_0, \ldots$ are identical the initial phase enters the control problem since the control amplitude $K_n$ is time dependent.

For the theoretical analysis we resort to linear stability analysis. For that purpose we rewrite eq.(1) as a first order difference equation of higher dimension. Introducing

‡ In our model the time–dependent modulation does not matter for fixed points. Only the time average will determine the stability.

$\underline{z}_n = (z_n^{(0)}, z_n^{(1)}, z_n^{(2)}) = (x_n, x_{n-1}, x_{n-2})$ eq.(1) reads

$$\underline{z}_{n+1} = F_{K_n}(\underline{z}_n) = \begin{pmatrix} f_{K_n}\left(z_n^{(0)}, z_n^{(2)}\right) \\ z_n^{(0)} \\ z_n^{(1)} \end{pmatrix} \tag{2}$$

where

$$f_K(x, x') = 1 - \mu x^2 + K(x - x') \tag{3}$$

denotes the right hand side of eq.(1). The period–two orbit $\underline{z}_0^* = (x_0^*, x_1^*, x_0^*)$, $\underline{z}_1^* = (x_1^*, x_0^*, x_1^*)$ is of course determined by the fixed point of the second iterate of the map (2), where the two different choices $F_{K_1}(F_{K_0}(\underline{z}))$ and $F_{K_0}(F_{K_1}(\underline{z}))$ correspond to opposite phases between the orbit and the control loop. Thus the stability properties of the orbit may depend on the phase. It is of course sufficient to analyse one case, e.g. $F_{K_1}(F_{K_0}(\underline{z}))$, since the other choice is just obtained by interchanging the values of $K_0$ and $K_1$ in the final result.

Linearision of the equation of motion (2) results in the Jacobian matrix

$$D(F_{K_1} \circ F_{K_0})(\underline{z}_0^*) = \begin{pmatrix} \partial_1 f_{K_1} \partial_1 f_{K_0} & \partial_2 f_{K_1} & \partial_1 f_{K_1} \partial_2 f_{K_0} \\ \partial_1 f_{K_0} & 0 & \partial_2 f_{K_0} \\ 1 & 0 & 0 \end{pmatrix} \tag{4}$$

where

$$\begin{aligned} \partial_\ell f_{K_1} &:= \partial_\ell f_{K_1}(f_{K_0}(x_0^*, x_0^*), x_1^*) = \partial_\ell f_{K_1}(x_1^*, x_1^*) \\ \partial_\ell f_{K_0} &:= \partial_\ell f_{K_0}(x_0^*, x_0^*) \end{aligned} \tag{5}$$

and $\partial_\ell$ denotes the derivative with respect to the $\ell^{th}$ argument. The characteristic equation determining the stability of the target state is obtained from the matrix (4) as

$$\begin{aligned} 0 &= \lambda^3 - \lambda^2(-2\mu x_1^* + K_1)(-2\mu x_0^* + K_0) \\ &\quad + \lambda(K_0(-2\mu x_1^* + K_1) + K_1(-2\mu x_0^* + K_0)) - K_0 K_1 \\ &= \lambda^3 + \lambda^2(-1 + K_0 + K_1 - K_0 K_1 + z(K_1 - K_0) + z^2) \\ &\quad + \lambda(-K_0 - K_1 + 2K_0 K_1 + z(K_0 - K_1)) - K_0 K_1 \end{aligned} \tag{6}$$

where the abbreviation $z := \sqrt{4\mu - 3}$ contains the dependence on the parameter of the map. Obviously the characteristic equation is asymmetric in $K_0$ and $K_1$ reflecting the possible preference of a particular phase. For successful control all eigenvalues $\lambda$ must be smaller than one in modulus. The Jury criterion (cf. appendix) yields necessary and sufficient conditions for stability.

## 3. Linear stability analysis of the control problem

Instead of using the control amplitudes $K_0$ and $K_1$ it is more convenient to introduce the mean value and the amplitude of the modulation

$$\kappa = \frac{1}{2}(K_0 + K_1), \quad \Delta = \frac{1}{2}(K_1 - K_0) \quad . \tag{7}$$

Evaluating the stability using eq.(6) and the Jury criterion (cf. appendix) is straightforward. Before dwelling on the general case we first summarise the results for the original time–delayed feedback scheme without modulation.

For $K_0 = K_1 = K$, i.e $\Delta = 0$ and $\kappa = K$ the saddle node criterion (A.2) yields $z^2 = 4\mu - 3 > 0$ which is always satisfied for parameter values where the period–two orbit exists. The criterion indicating period doubling bifurcations, eq.(A.3), reads

$$z^2 = 4\mu - 3 < (2K - 1)^2 + 1 \quad . \tag{8}$$

The first Hopf criterion (A.4) yields the constraint $K^2 < 1$. Thus the control amplitude may be limited from the very beginning to the domain $|K| < 1$. The second Hopf criterion (A.5) yielding the two inequalities (A.6) and (A.7) finally leads to the constraints

$$z^2 = 4\mu - 3 < (2K + 1)(1 - K^2)/K^2 \tag{9}$$

and

$$z^2 = 4\mu - 3 > -(2K^2 - 2K + 1)(1 - K^2)/K^2 \quad . \tag{10}$$

Since the right hand side of eq.(10) is negative in view of the constraint $|K| < 1$, the inequality (10) does not yield a condition for the control domain. Thus we just need to consider the inequalities (8), (9), $|K| < 1$, and $z^2 = 4\mu - 3 > 0$. Figure 1 summarises the results of our analysis. We obtain a "lower" control threshold caused by a period doubling bifurcation and an "upper" threshold through a Hopf instability. The control domain shows for $z^2 > 2$ the typical triangular shape (cf. Figure 1, the part of the hatched area above the dotted line) which is known from general considerations on time–delayed feedback control [9]. In particular, control fails if the Lyapunov exponent of the unstable orbit $\ln |z^2 - 1|$ becomes too large.

Let us now investigate how the control properties change when a modulation of finite amplitude $\Delta$ is applied. For that purpose we evaluate the stability criteria (A.2), (A.3), (A.4), (A.6), and (A.7) for the polynomial (6). The condition on the saddle node bifurcation (A.2) again results in $z^2 = 4\mu - 3 > 0$. The period doubling constraint (A.3) yields (cf. eq.(8))

$$(z + 2\Delta)^2 < (2\kappa - 1)^2 + 1 \quad . \tag{11}$$

The first Hopf condition (A.4) leads to

$$(\kappa^2 - \Delta^2)^2 < 1 \quad . \tag{12}$$

Finally the second type of Hopf constraint, eqs.(A.6) and (A.7), results in (cf. eq.(9))

$$(\kappa^2 - \Delta^2)z^2 - 2\Delta(1 - \kappa^2 + \Delta^2)z < (2\kappa + 1)(1 - \kappa^2 + \Delta^2) \tag{13}$$

and (cf. eq.(10))

$$\begin{aligned}(\kappa^2 - \Delta^2)z^2 - 2\Delta(1 - \kappa^2 + \Delta^2)z > & -(2\kappa^2 - 2\kappa + 1 - 2\Delta^2) \\ & \times (1 - \kappa^2 + \Delta^2) \quad .\end{aligned} \tag{14}$$

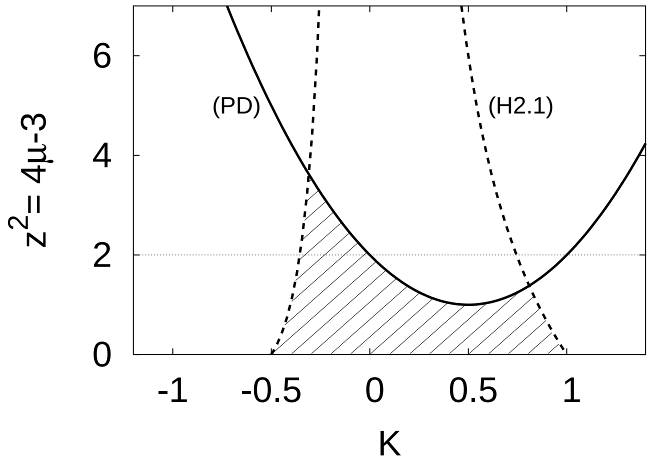


**Figure 1.** Control domain for plain time–delayed feedback control of the period–two orbit of the logistic map. (PD) (full line): boundary caused by period doubling (cf. eq.(8)), (H2.1) (broken line): boundary caused by Hopf instability (cf. eq.(9)). The hatched area indicates the stable domain. For parameter values $z^2 < 2$ (i.e. below the dotted line) there is no need for control since the free system $K = 0$ already has a stable period–two orbit.

It is not easy to spot the influence of the modulation $\Delta$ on the stability diagram. Concerning the period doubling threshold (11) the modulation shifts the boundary in vertical direction so that for negative $\Delta$ an increase of control performance is expected. We recall that different signs of $\Delta$ just correspond to the two different phase lags between the orbit and the control loop. For the boundaries generated by Hopf instabilities no such simple dependence can be obtained from eqs.(13) and (14). Figure 2 contains control diagrams for different values of the modulation amplitude. As in the case without modulation only the conditions (11) and (13) determine the control boundaries. Negative amplitudes increase the control domain till finally the period doubling constraint disentangles from the Hopf condition at about $\Delta \approx -0.6$. For smaller $\Delta$ values the control domain decreases again. An additional boundary appears at small values of $z^2 = 4\mu - 3$ which is caused by the second root of the period doubling constraint (11). This boundary shifts upwards if $\Delta$ is decreased. In conclusion, there exists an optimal value for the modulation amplitude. Above all a remarkable change of the size of the control domain is observed for $\Delta$ values with different sign expressing the preference of a particular phase lag.

## 4. Conclusion

As shown by the previous analysis modulated time–delayed feedback control enhances the control performance of autonomous systems. As time–dependent modulation breaks the time translation invariance certain phase lags between the periodic orbit and the controller are selected due to enhanced stability properties. In our particular model the

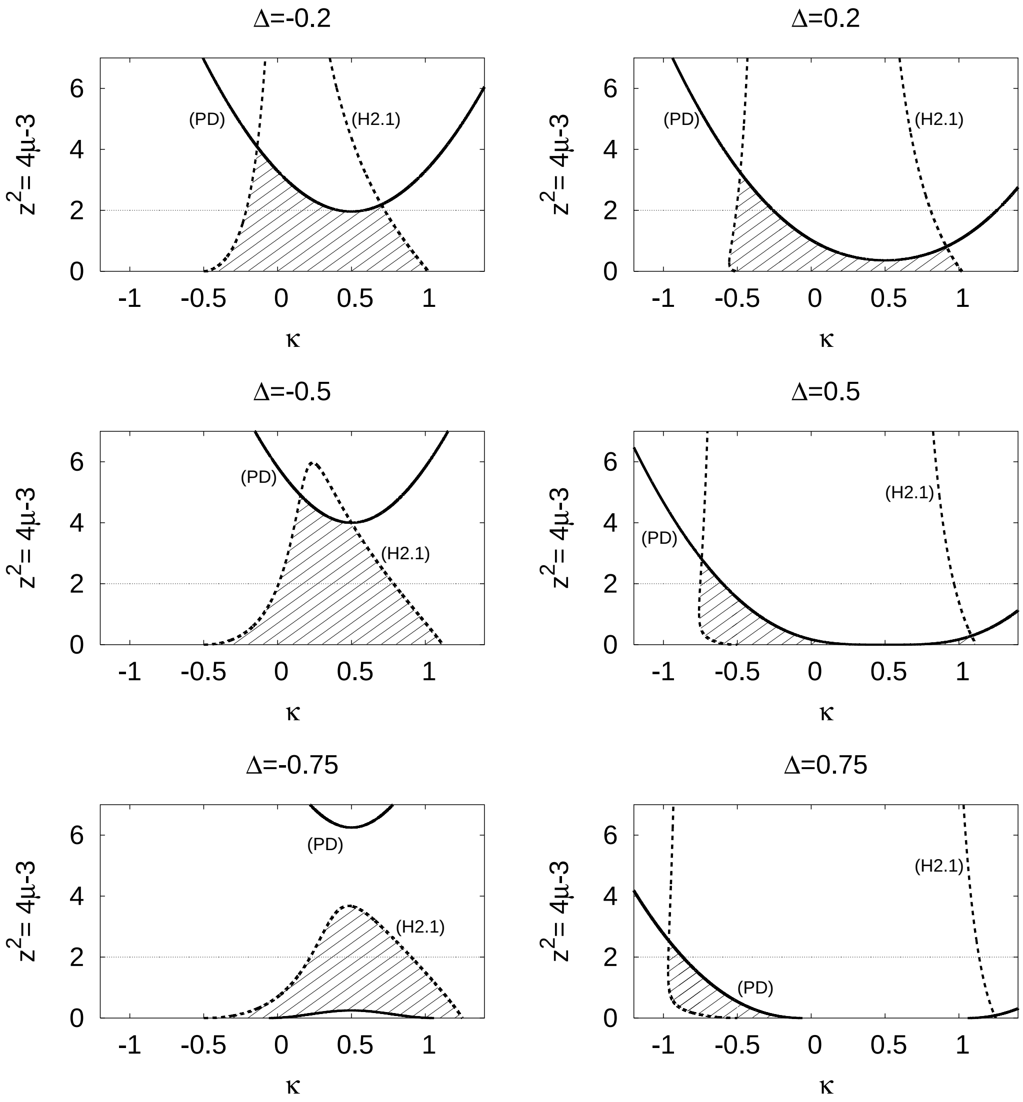


**Figure 2.** Control domain for modulated time–delayed feedback control of the period–two orbit of the logistic map (cf. figure 1). (PD) (full line): boundary caused by period doubling (cf. eq.(11)), (H2.1) (broken line): boundary caused by Hopf instability (cf. eq.(13)). The hatched area indicates the stable domain. Parameter values $z^2 < 2$ (i.e. below the dotted line) yield stable period–two orbits even without control. Plots in the same row correspond to opposite phase lags.

period–two orbit, which according to the plain time–delayed feedback method can be stabilised in a limited parameter range only, becomes accessible for control in much larger parameter domains. As a modulation of control parameters is trivial to implement in experimental setups the method looks promising from the point of view of applications.

Our analysis can be carried out for higher periodic orbits as well. We expect a similar influence of phase lags on the control performance but the full analytical discussion becomes quite tedious even if one sticks to simple one–dimensional maps. In addition, one has to keep in mind that control of orbits with high periods may call for extended control schemes [3] because of the limitations mentioned above.

Modulated control selects a particular phase lag of the target state. Thus the control method may be also feasible for phase locking and enhancement of synchronisation between different subsystems. Of course such advanced topics require the investigation of model systems which are far beyond our simple toy example.

We expect that our results show some generic features of modulated time–delayed feedback control. As the influence of modulated control already shows up in first order when a perturbation expansion in terms of the modulation amplitude is applied§, the sign of the modulation amplitude matters. Thus the control performance is either suppressed or enhanced depending on the sign of the phase lag between the orbit and the controller. But of course general properties of the control scheme including global features are at the current stage difficult to predict.

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## Appendix: Jury criterion

For the linear stability of a general time–discrete dynamical system all the roots of the characteristic equation

$$P(\lambda) = a_0 + a_1\lambda + \cdots + a_N\lambda^N = 0 \tag{A.1}$$

must be contained in the unit circle, $|\lambda| < 1$. Decades ago a necessary and sufficient criterion for such a condition has been derived by Schur and Cohn [10] and then further simplified by Jury [11]. It is quite well established in the engineering context but hardly referenced in the physics literature. In fact, the conditions on the coefficients for general $N$ are quite cumbersome and we here just confine to the case $N = 3$. Assuming without loss of generality $a_3 > 0$ the Jury criterion reads

$$(\mathrm{SN}): \quad P(1) = a_0 + a_1 + a_2 + a_3 > 0 \tag{A.2}$$

$$(\mathrm{PD}): \quad P(-1) = a_0 - a_1 + a_2 - a_3 < 0 \tag{A.3}$$

$$(\mathrm{H1}): \quad a_3 > |a_0| \tag{A.4}$$

$$(\mathrm{H2}): \quad |a_3^2 - a_0^2| > |a_1 a_3 - a_0 a_2| \quad . \tag{A.5}$$

If one looks at the borderline cases, i.e. when one of the inequalities becomes an identity, then an instability in the corresponding dynamical system takes place. Condition (SN) implies that a root $\lambda = 1$ appears, i.e. a saddle node bifurcation happens, whereas for (PD) $\lambda = -1$ solves the characteristic equation causing a period doubling bifurcation. The two remaining cases (H1) and (H2) each correspond to a complex conjugated pair

§ The first order vanishes when some symmetry properties are met like e.g. for control through eigenmodes (cf. [7]).

on the unit circle indicating a Hopf bifurcation. Condition (A.5) may be simplified further taking the inequality (A.4) into account.

$$\text{(H2.1)}: \quad a_3^2 - a_0^2 > a_1 a_3 - a_0 a_2 \tag{A.6}$$

$$\text{(H2.2)}: \quad a_3^2 - a_0^2 > a_0 a_2 - a_1 a_3 \tag{A.7}$$

There exists a simple link between the Jury criterion and the Hurwitz criterion [12] which is more common in the physics literature and which applies to time continuous systems. If

$$Q(\sigma) = b_0 \sigma^N + b_1 \sigma^{N-1} + \cdots + b_N \tag{A.8}$$

denotes the characteristic polynomial where $b_0 > 0$, then the necessary and sufficient condition for all roots $\sigma$ having negative real part can be expressed in terms of inequalities for determinants

$$H_1 = b_1 > 0, \quad H_2 = \begin{vmatrix} b_1 & b_3 \\ b_0 & b_2 \end{vmatrix} > 0, \quad H_3 = \begin{vmatrix} b_1 & b_3 & b_5 \\ b_0 & b_2 & b_4 \\ 0 & b_1 & b_3 \end{vmatrix} > 0, \ldots \quad H_N > 0 \tag{A.9}$$

where all matrix elements $b_{n>N}$ have to be replaced by zero. The Hurwitz criterion is much simpler to formulate for general $N$ than the Jury criterion. In fact, both criteria can be related to each other. Noting that

$$\lambda = (1+\sigma)/(1-\sigma) \tag{A.10}$$

yields a conformal mapping from the left half complex plane to the inner of the unit circle we can rewrite eq.(A.1) as a polynomial to which the Hurwitz criterion can be applied

$$\begin{aligned} 0 &= (1-\sigma)^N P\left(\frac{1+\sigma}{1-\sigma}\right) \\ &= a_0(1-\sigma)^N + a_1(1-\sigma)^{N-1}(1+\sigma) + \cdots + a_N(1+\sigma)^N \quad . \end{aligned} \tag{A.11}$$

But even this approach becomes quite cumbersome for general $N$.


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