Mean–field theory of critical coupled map lattices

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Abstract. We study the single–site approximation of the Perron–Frobenius equation for a coupled map lattice exhibiting a phase transition at a critical value g_c of the coupling constant. We found that the critical exponents are the same as in the usual mean–field theory of equilibrium statistical mechanics. Remarkably, the value of g_c is within six percent of the one previously obtained by numerical simulations with asynchronous updating.

PACS numbers: 05.45.+b, 05.70.Jk

Short title: Mean–field theory of coupled map lattices

June 10, 1998

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1. Introduction

Critical phenomena in extended dynamical systems attracted considerable interest recently. In particular, coupled map lattices were introduced as paradigmatic models of nonequilibrium systems undergoing (in the large–size limit) a second–order phase transition between two chaotic states [1, 2]. In those models such a transition is associated with a breaking of the Ising symmetry, and is thus expected to share the same static critical exponents of the Ising model itself. However, numerical simulations [3] show that this is not the case. More precisely, the critical exponent of the correlation length for the model of Ref. [2] turned out to be significatively different from the Ising value. Moreover, the latter are recovered as soon as the updating rule is changed from parallel to sequential, thus indicating that some features of the microscopic dynamics may be relevant for the critical behaviour. Besides of those facts, it is not completely clear why a phase transition behaviour emerges at all and why the coupling is effectively "ferromagnetic".

For all the above reasons, even the simplest analytical approach, namely the mean–field like approximation, is worth to be investigated. Actually, it is not clear to which extent it can be applied in nonequilibrium cases and which properties it may share with its equilibrium counterpart. Moreover, it is still questionable that such an approach is able to reproduce the qualitative behaviour of coupled map lattices in general.

We will discuss such problems for the coupled map lattice introduced in Ref. [2]. Its equations of motion read as

$$x_{n+1}^{(\nu)} = (1-gd)f(x_n^{(\nu)}) + g \sum_{\mu\in\mathcal{U}_\nu} f(x_n^{(\mu)}) \quad , \tag{1}$$

where $x_n^{(\nu)}$ is the field variable at time n and the index ν enumerates the sites on a square lattice. Here, $\mathcal{U}_\nu$ denotes the set of the d neighbours to which the νth site is coupled and g is the coupling constant. Rather than considering the original isotropic nearest–neighbour coupling on a square lattice we will refer to its modification presented in [3], which couples four instead of five sites. This choice has the advantage of simplifying the algebraic calculations and offers the possibility to compare our results with the numerical simulations reported in [3]. Since we employ some mean–field theory, our approach applies regardless of the special type of spatial geometry and boundary conditions do not play a significant role.

The local map, which is defined on the interval $[-1,1]$

$$f(x) = \begin{cases} -3x-2 & \text{for } -1 \le x \le -1/3 \\ 3x & \text{for } -1/3 < x < 1/3 \\ -3x+2 & \text{for } 1/3 \le x \le 1 \end{cases} \tag{2}$$

obeys an Ising like symmetry $f(x) = -f(-x)$. In the supercritical region $g > g_c$ the symmetry is spontaneously broken and the coupled map lattice exhibits two ordered phases, characterised by a non–vanishing value of the "magnetisation" $\langle \sum_\nu x^{(\nu)} \rangle$.

2. Mean–field approach

Our starting point is the evolution equation for the single–site probability density $\rho(x)$ (cf. e. g. [4]), as usually obtained by projection of the Perron–Frobenius equation for the full probability density. The single–site density is expressed in terms of the four–sites joint probability density $\rho^{(4)}$ as

$$\rho_{n+1}(x) = \int dy_0 dy_1 dy_2 dy_3 \delta \left[x - T(y_0, y_1, y_2, y_3) \right] \rho_n^{(4)}(y_0, y_1, y_2, y_3) \quad . \tag{3}$$

Here we have introduced the abbreviation

$$T(y_0, y_1, y_2, y_3) = (1 - 3g) f(y_0) + g \left[f(y_1) + f(y_2) + f(y_3) \right] \quad . \tag{4}$$

The function T is invariant under all permutations of the variables y_1, y_2, y_3. The mean–field Perron–Frobenius equation [5] is as usual obtained by neglecting multiple correlations, namely by letting

$$\rho_n^{(4)}(y_0, y_1, y_2, y_3) \;\rightarrow\; \rho_n(y_0)\rho_n(y_1)\rho_n(y_2)\rho_n(y_3) \quad . \tag{5}$$

Then eq.(3) becomes a nonlinear integral equation

$$\rho_{n+1}(x) = \int dy_0 dy_1 dy_2 dy_3 \delta \left[x - T(y_0, y_1, y_2, y_3) \right] \rho_n(y_0)\rho_n(y_1)\rho_n(y_2)\rho_n(y_3) \quad . \tag{6}$$

It should be stressed that the approximation (5) leads to different results if one chooses different coordinate systems in the full phase space of the map lattice. In fact, neglecting correlations has a different meaning in different coordinate systems. Hence, similar to statistical mechanics, it makes no sense to speak about *the* mean–field approximation. The formulation chosen here seems to be quite appropriate for the analytical calculation.

Since we are interested in stationary properties we look for the fixed point solution $\rho_n = \rho_*$. In the paramagnetic region, i. e. for sufficiently small coupling, the solution is symmetric $\rho_*(x) = \rho_*(-x)$ and does not give rise to a finite magnetisation, namely $\int dx\, x \rho_*(x) = 0$. At the critical coupling g_c such solution will get unstable in favour of a non–symmetric one. In order to tackle this problem we first solve for the symmetric stationary solution employing a Fourier series expansion

$$\rho_*(x) = \frac{1}{\sqrt{2}} \sum_{k=-\infty}^{+\infty} c_k \, \exp(i\pi k x) \quad . \tag{7}$$

Here, because of normalisation $c_0 = 1/\sqrt{2}$, and all the coefficients $c_k = c_{-k}$ are real because of symmetry. With the abbreviation

$$F_{kk'}(g) = \int \exp[i\pi(k'x - kgf(x))] \, dx$$

$$= 4\cos(2\pi k'/3)\,\frac{\sin[\pi(k'+3kg)/3]}{\pi(k'+3kg)} + 2\frac{\sin[\pi(k'-3kg)/3]}{\pi(k'-3kg)} \qquad (8)$$

the fixed point equation reads

$$c_k = \frac{1}{2^{5/2}}\left(\sum_{k'} F_{kk'}(1-3g)c_{k'}\right)\left(\sum_{k''} F_{kk''}(g)c_{k''}\right)^3 \quad . \qquad (9)$$

Although such an equation can be numerically solved by iterative methods, it is convenient to look for an approximate analytical solution. As $F_{kk'} + F_{k-k'} = 0$ holds whenever the index k' is not an integer multiple of three, the right hand side of equation (9) contains only Fourier coefficients of the form $c_{k'=3l}$ for symmetric densities. If we truncate the expansion (7) at the fifth mode, then the right hand side of eq.(9) only contains the coefficient c_3. The latter is self consistently determined by

$$\begin{aligned}\sqrt{2}c_3 = \frac{1}{4}&\left(F_{30}(1-3g) + [F_{33}(1-3g) + F_{3-3}(1-3g)]\sqrt{2}c_3\right)\\ &\left(F_{30}(g) + [F_{33}(g) + F_{3-3}(g)]\sqrt{2}c_3\right)^3 \quad . \qquad (10)\end{aligned}$$

The polynomial (10) admits two real solutions in the whole range $0 < g < 1/3$, but only one of them is smaller than $1/2$. The remaining Fourier coefficients up to order five are now obtained if we plug in the expansion (7) with the solution of eq.(10) into the right hand side of eq.(9). The analytical expression thus obtained coincides up to five digits with the full numerical solution of eq.(9) if many modes are taken into account.

We are now going to evaluate explicitly the critical point. It corresponds to the value of coupling where the symmetric solution loses its stability. Considering therefore small deviations from it $\rho_n = \rho_* + \delta\rho_n$, and expanding eq.(6) we obtain

$$\delta\rho_{n+1}(x) = (\mathcal{L}\delta\rho_n)(x) + (\mathcal{C}[\delta\rho_n,\delta\rho_n])(x) + (\mathcal{D}[\delta\rho_n,\delta\rho_n,\delta\rho_n])(x) + \cdots \quad .(11)$$

The deviations obey the constraint $\int dx\,\delta\rho_n = 0$ because of the normalisation condition. The stability properties are determined by the eigenvalue problem for the linear operator

$$(\mathcal{L}\delta\rho)(x) \;=\; \int dy\,\Lambda(x,y)\delta\rho(y) \quad . \qquad (12)$$

Its kernel reads

$$\begin{aligned}\Lambda(x,y) = \int &dz_1 dz_2 dz_3\; \left(\delta\left[x - T(y,z_1,z_2,z_3)\right] + 3\delta\left[x - T(z_1,y,z_2,z_3)\right]\right)\\ &\rho_*(z_1)\rho_*(z_2)\rho_*(z_3) \quad . \qquad (13)\end{aligned}$$

Under quite mild conditions on ρ_* (e. g. continuity is sufficient) the kernel is continuous. Hence the operator (12) is compact and its spectrum consists of isolated eigenvalues which accumulate at most at zero [6]. In addition, the bifurcation behaviour of the symmetric density is essentially identical to bifurcations in low–dimensional dynamical systems. In particular, the instability is typically caused by a single eigenvalue λ crossing the unit circle in the complex plane.

The numerical solution of the eigenvalue problem is accomplished by representing Λ on a truncated Fourier basis, and diagonalising the resulting finite–dimensional matrix. The spectrum thus obtained at the critical coupling is displayed in figure 1. An isolated eigenvalue attains the unit circle along the real axis and the corresponding critical eigenfunction $v(x)$ is odd with respect to the space inversion $v(x) = -v(-x)$ (cf. figure 2) †. Notice that, as the kernel (13) is not symmetric and the corresponding linear operator (12) is in general not selfadjoint, the critical left–eigenfunction $w(x)$ may differ from $v(x)$. From the numerical diagonalisation of the matrix the critical coupling is

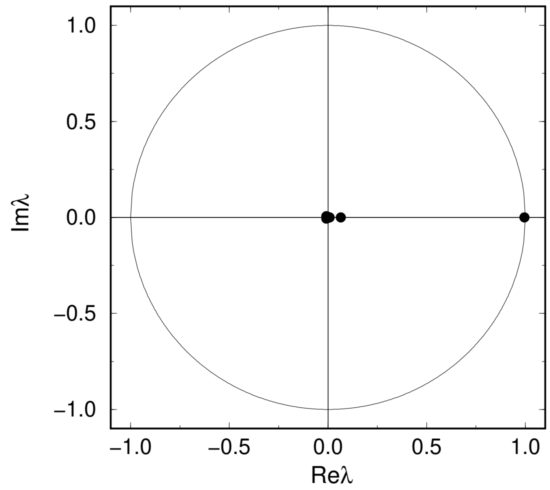

Figure 1. Spectrum of the operator (12) at the critical coupling g_c.

very accurately determined to be $g_c = 0.1496\ldots$, which remarkably deviates only 6% from the value $0.1584\ldots$ of the critical point obtained in numerical simulations of the full map lattice with asynchronous updating [3]. This coincidence is consistent with the idea that the latter dynamic rule induces less spatial correlations than the synchronous one.

3. Bifurcation analysis

The bifurcation scenario described at the end of the preceding section is of the pitchfork type. Hence the growth of the critical mode beyond the instability is expected to scale as

† Since the full equation (6) is homogeneous of order four the linear operator (12) admits a Goldstone–like mode with eigenvalue $\lambda = 4$ and eigenfunction $\rho_*(x)$.

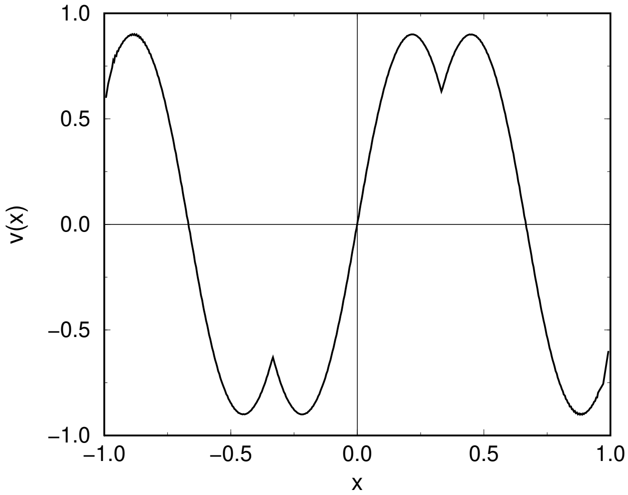

Figure 2. The critical eigenfunction $v(x)$ corresponding to $\lambda = 1$.

$(g - g_c)^{1/2}$. We stress that such a critical behaviour follows solely from the compactness of the linear operator. Therefore, we expect that every mean–field approximation of the Perron–Frobenius equation shares the same critical exponents.

To support such general remarks we are now going to perform a complete normal form reduction of the full mean–field equation. For that purpose one needs the quadratic and cubic contributions of eq.(11) which are given by

$$(\mathcal{C}[\varphi,\psi])(x) = \int dy_1 dy_2 \Gamma(x,y_1,y_2)\varphi(y_1)\psi(y_2) \tag{14}$$

$$(\mathcal{D}[\varphi,\psi,\omega])(x) = \int dy_1 dy_2 dy_3 \Delta(x,y_1,y_2,y_3)\varphi(y_1)\psi(y_2)\omega(y_3) \quad . \tag{15}$$

The corresponding kernels read for example

$$\Gamma(x,y_1,y_2) = \int dz_1 dz_2 \left[3\delta(x - T(y_1,y_2,z_1,z_2)) + 3\delta(x - T(z_1,y_1,y_2,z_2))\right] \rho_*(z_1)\rho_*(z_2) \tag{16}$$

$$\Delta(x,y_1,y_2,y_3) = \int dz \left[\delta(x - T(z,y_1,y_2,y_3)) + 3\delta(x - T(y_1,z,y_2,y_3))\right]\rho_*(z) \quad . \tag{17}$$

The centre manifold tangential to $v(x)$ is expanded as

$$\delta\rho_n(x) = \alpha_n v(x) + \alpha_n^2 r(x) + \alpha_n^3 s(x) + \cdots \quad , \tag{18}$$

where the scalar α_n denotes the coordinate on the one dimensional manifold. To remove the ambiguity on the transversal vectors $r(x)$ and $s(x)$, we chose $\langle w|r\rangle = 0$, $\langle w|s\rangle = 0$

with respect to the canonical bilinear form $\langle\psi|\varphi\rangle = \int dx\, \psi(x)\varphi(x)$. The time evolution on the centre manifold obeys

$$\alpha_{n+1} = A\alpha_n + B\alpha_n^2 + C\alpha_n^3 + \cdots \quad . \tag{19}$$

If we now plug in eqs.(18) and (19) into eq.(11) and compare the different powers in α_n, then the unknown expansion coefficients in eq.(19) will be fixed. To the first order we just obtain the eigenvalue equation

$$(\mathcal{L}v)(x) = Av(x) \tag{20}$$

so that $A = \lambda$. To the second order we obtain a linear inhomogeneous equation determining the transversal vector r

$$A^2 r(x) - (\mathcal{L}r)(x) = (\mathcal{C}[v,v])(x) - Bv(x) \quad . \tag{21}$$

Since the operator on the left hand side becomes singular at the critical point $g = g_c$, but the solution has to stay regular, the right hand side must obey a Fredholm condition at $g = g_c$. In particular, the right hand side has to be orthogonal to $w(x)$. Moreover, as the first term on the right hand side drops from this condition by symmetry, we are left with $B = 0$. Finally, at the third order we obtain

$$A^3 s(x) - (\mathcal{L}s)(x) = (\mathcal{C}[v,r])(x) + (\mathcal{C}[r,v])(x) + (\mathcal{D}[v,v,v])(x) - Cv(x) \quad . \tag{22}$$

The Fredholm condition ensuring a regular solution at criticality determines the cubic coefficient as

$$C\langle w|v\rangle = \langle w|\mathcal{C}[v,r] + \mathcal{C}[r,v] + \mathcal{D}[v,v,v]\rangle \quad . \tag{23}$$

Eq.(23) is easily evaluated using eqs.(14), (15), and (21). If the representation in terms of Fourier modes is employed, then all matrix elements can be expressed in terms of (8). Of course the modulus of C has no special meaning, since it depends on the normalisation of $v(x)$. We just chose $\int dx\, v^2(x) = 1$ and obtain $C = -8.889 \times 10^{-3}$. Hence, the pitchfork bifurcation is supercritical as already mentioned above. The stationary distribution in the supercritical region is thus readily evaluated from the normal form (19) as $\rho_*(x) + \sqrt{(\lambda-1)/(-C)}\, v(x)$, where ρ_* is the (symmetric) stationary density at the critical point g_c. Accordingly, the critical mode and the magnetisation grow like $(\lambda-1)^{1/2} \simeq (g-g_c)^{1/2}$ beyond criticality. In addition, it is important to notice that the system is close to a super–subcritical transition. In fact, if one considers the definition (23) for arbitrary coupling g, then a change in the sign of C occurs slightly below the transition point g_c at $g = 0.1477\ldots$. For that reason, the correct scaling behaviour occurs only in a narrow region beyond g_c, and the evaluation of the critical exponents from a numerical solution of eq.(6) is almost impossible to perform. The last point again emphasises the importance of our analytical results.

Sofar we have dealt with the critical behaviour of the order parameter only. Let us discuss now the counterpart of the static susceptibility in equilibrium systems. In analogy with the latter, we may tentatively define it as the derivative of the order parameter with respect to a suitable "symmetry breaking field". This amounts to introduce some external parameter h spoiling the symmetry of the original single site map. Although not unique, a natural choice is for example to replace f in eq.(1) with

$$f_h(x) = \begin{cases} -3x-2 & \text{for } -1 \leq x \leq -1/3 \\ 3x & \text{for } -1/3 < x < 1/3 \\ -3(1+h)x+2-h & \text{for } 1/3 \leq x \leq 1 \end{cases} \quad . \tag{24}$$

Here the transition rate from $[0,1]$ to $[-1,0]$ is lowered upon increasing the parameter h, thus mimicking the effect of a static magnetic field in the Ising system. The mean–field susceptibility can then be defined as

$$\chi := \left.\frac{\partial\langle x\rangle}{\partial h}\right|_{h=0} = \left.\frac{\partial}{\partial h}\int dx\, x\, \rho_*(x)\right|_{h=0} \quad , \tag{25}$$

where $\rho_*(x)$ denotes now the fixed point solution of eq.(6) with the single site map (24). If we suppose that such a solution depends on h in a differentiable way, we can take the formal derivative of eq.(6) obtaining

$$\begin{aligned}(1-\mathcal{L})\left.\frac{\partial\rho_*}{\partial h}\right|_{h=0}(x) = &-\int dy_0dy_1dy_2dy_3\delta'[x-T_{h=0}(y_0,y_1,y_2,y_3)] \\ &\left.\frac{\partial T_h(y_0,y_1,y_2,y_3)}{\partial h}\right|_{h=0}\rho_*(y_0)\rho_*(y_1)\rho_*(y_2)\rho_*(y_3) \quad ,\end{aligned} \tag{26}$$

where the definition of (12) has been also taken into account. In order to solve this equation for $\partial\rho_*/\partial h|_{h=0}$, we expand both the latter quantity and the the right hand side of eq.(26) in terms of eigenmodes of $\mathcal{L}$. As only one eigenvalue crosses the unit circle and the right hand side of eq.(26) remains bounded under quite mild conditions on ρ_*, all the coefficients of such expansion remain bounded in the vicinity of the critical point, except for the one of the critical mode $v(x)$. This coefficient develops a singularity of the form $|\lambda-1|^{-1}$, which carries over to $\partial\rho_*/\partial h|_{h=0}$, provided that the expansion in terms of eigenmodes converges absolutely.

The same result can of course be derived along the lines of bifurcation theory. Since the instability without the symmetry breaking field is governed by the pitchfork normal form and only a single eigenvalue becomes critical, one expects from the very beginning that the symmetry breaking unfolds the normal form to the cusp case (cf. [7]). That indeed occurs if one follows the normal form reduction presented above. Altogether, we can conclude that the static susceptibility diverges as $|g-g_c|^{-1}$, in accordance with the simple mean–field theory for equilibrium systems.

4. Conclusions

We have shown that a mean–field approach for the Miller–Huse model reproduces the critical behaviour known from simple mean–field approximations in equilibrium statistical mechanics. The value of the critical exponents basically originates from the compactness property of the linearised operator. Since such a property holds quite generally, our result is supposed to apply for a large class of coupled map lattices and almost all kinds of mean–field approximations. Corrections to mean–field scaling can only come from a continuous spectrum of the full Perron–Frobenius operator, similar to findings in equilibrium statistical mechanics, e. g. for the two dimensional Ising model. The coincidence with equilibrium mean–field theories is far from being obvious. For instance, globally coupled maps behave quite different compared to the mean–field approximation of the type employed here (cf. [8] and references therein). That observation is in striking contrast to equilibrium statistical mechanics, where mean–field approaches and long–range coupled models are often equivalent.

As mentioned in the beginning, the critical behaviour of the full map lattice depends on the updating rule. In particular, the sole occurrence of equilibrium Ising exponents for asynchronous updating was attributed to a kind of destruction of coherence by the updating rule. Of course, the plain mean–field approach cannot explain such differences in critical exponents at all. Nevertheless, the mentioned interpretation is fully consistent with the fact that the mean–field approach (which neglects all correlations) yields a good estimate of g_c for the model with asynchronous updating.

At equilibrium, the static susceptibility can be expressed in terms of the spatial correlation function by virtue of the properties of the canonical distribution. This is of course no longer true for out of equilibrium systems as the coupled map lattices. Hence, it would be tempting to check whether even in this case the critical behaviour of the susceptibility coincides with that of spatial correlations. Such a coincidence, which of course requires quite accurate numerics, would indicate a relation between response and correlations in nonequilibrium systems too.

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