# On the eigenvalue spectrum for time–delayed Floquet problems

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**Abstract**

A linear homogeneous scalar differential–difference equation with harmonic time dependence is investigated. The associated eigenvalue problem is solved in terms of a continued fraction expansion for the characteristic equation. The dependence of the largest eigenvalue on the system parameters, being relevant for stability of periodic states in delay systems, is discussed in detail. The competition between the two time scales, the delay and the external period cause intricate structures. The result suggests features to improve control of chaos by time–delayed feedback schemes with time–dependent control amplitudes.

*Keywords:* Stability analysis, Differential–difference equation, Chaos control

## 1 Introduction

The dynamics of delay systems has received considerable interest during the last few years since time delay either may be an inherent property of the dynamics as demonstrated by classical examples like laser systems [1] or biological models [2], or time delayed signals can be efficiently used for control processes [3]. From the pure theoretical point of view delay systems constitute the simplest class of equations having potentially high dimensional dynamical behaviour [4]. Despite several thorough numerical investigations of certain model systems less is known from the analytical point of view apart from general existence and uniqueness theorems. Such a statement holds even for linear time–dependent equations which occur frequently in stability investigations of

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periodic states. It is the aim of this article to study the properties of the scalar differential–difference equation

$$\dot{x}(t) = cx(t) + B(t)x(t-\tau) \quad , \tag{1}$$

where the coefficient $B$ is periodic in time $B(t) = B(t+T)$. Later on we will concentrate on the special harmonic choice

$$B(t) = 2b\cos(\omega t)\,, \quad \omega = \frac{2\pi}{T} \quad . \tag{2}$$

At the very beginning one might wonder why such a simple and special equation like eq.(1) is of some interest and why it has some relation to nonlinear dynamics at all. Let me first recall in more detail the motivation for the subsequent study. If one investigates stability properties of periodic orbits one usually ends up with equations quite similar to eq.(1). To be more definite consider the Pyragas scheme for chaos control [5] which in terms of equations of motion is given by

$$\dot{z}(t) = f\left(z(t), K\left\{g\left[z(t)\right] - g\left[z(t-\tau)\right]\right\}\right) \quad . \tag{3}$$

Here $z$ denotes the internal degrees of freedom and $g[z]$ some measurable scalar quantity. The second argument of the right hand side takes into account that the time–delayed difference of the measured signal is feed back to the dynamical system with a control amplitude $K$. If one adjusts the delay time $\tau$ according to the period of a periodic orbit $\zeta(t) = \zeta(t+\tau)$ of the uncontrolled system ($K = 0$), then stability is determined by the evolution of small deviations from that orbit

$$\begin{aligned}\delta\dot{z}(t) = &\, D_1 f\left(\zeta(t), 0\right)\delta z(t) \\ &+ K d_2 f\left(\zeta(t), 0\right)\left(dg\left[\zeta(t)\right]\left\{\delta z(t) - \delta z(t-\tau)\right\}\right) \quad .\end{aligned} \tag{4}$$

One may further cast eq.(4) in a vector type generalisation of eq.(1) by shifting the time dependencies solely to the delay term by virtue of a Floquet transformation (appendix A.1). It is however evident that within the present motivation the period of the time–dependent coefficients and the delay time are connected with each other, at least if the control amplitude is time–independent. In such cases it is well established that some severe constraint on the stability of the solution occurs [6,7] which is related to the torsion of the free orbit and which causes some rigidity on the associated eigenvalue spectrum. Such constraints might be relaxed by applying a time–dependent control amplitude [8,9]. The additional time scale decouples the periodicity of the coefficients from the delay time and one may expect that the solution properties of eq.(1) may strongly change if the period of the coefficient and the delay time are varied independently. It is precisely this subject we focus on in what follows.

Let us now forget about the preceding motivation and come back to eq.(1). The quite general theory of functional differential equations [4] tells us that we can decompose the full solutions in eigenmodes using the usual Floquet decomposition. Then eq.(1) reduces to the eigenvalue problem

$$\lambda q(t) + \dot{q}(t) = cq(t) + B(t)e^{-\lambda\tau}q(t-\tau), \quad q(t) = q(t+T) \quad . \tag{5}$$

The different eigenmodes do not build up a complete solution of eq.(1) but may miss the so called small solutions which decay faster than exponentially. Fortunately, such a failure does not matter from the point of view of stability analysis. Furthermore, the general theory tells us that all eigenvalues are isolated, mainly because of the compactness properties of integral operators. Hence, the eigenvalue problem is quite well suited from the mathematical point of view.

As already indicated there are certain cases which are relatively simple to analyse, namely if $B$ is time–independent [10] or if the delay is an integer multiple of the period. Then the whole delay problem can be reduced to ordinary differential equations [4] and global properties of the spectrum can be obtained from contour integrals (cf. [11]). In the scalar case even a reduction to algebraic equations is possible since the delay drops from eq.(5) and plain integration leads to

$$\lambda = c + \bar{B}e^{-\lambda\tau} \quad . \tag{6}$$

Here $\bar{B}$ denotes the time average. The analysis of such algebraic eigenvalue problems dates back at least half a century [12] and systematic investigations can be found in the literature [10].

Much less is known if the ratio between delay $\tau$ and period $T$ is non–integer. The preceding comments on control properties indicate that new feature may enter by virtue of the additional time scale. In principle one might try to proceed by reducing eq.(5) to a system of ordinary differential equations. The step can be accomplished if the ratio of the period and the delay is rational. Some insight might be gained from such a procedure (cf. appendix A.2) but no systematic approach seems to be possible along this lines. Here we follow a different route. For reasons of simplicity the subsequent discussion is restricted to the special choice (2).

## 2 Analytical properties of the characteristic equation

Even from the numerical point of view it does not seem to be quite straightforward to solve the eigenvalue problem (5), (2) by direct integration. An expansion of the eigenfunction in terms of Fourier modes seems to be appropriate and in fact, using a standard procedure the Fourier coefficients can be eliminated [13]. A closed expression for the characteristic equation is obtained in terms of a continued fraction expansion. With suitable non–dimensional abbreviations for the eigenvalue, the rescaled driving amplitude, and the ratio between delay and period

$$z := \frac{\lambda - c}{\omega}, \quad \beta := \frac{b}{\omega} e^{-c\delta}, \quad \delta := \omega\tau \tag{7}$$

the characteristic equation reads (cf. appendix B)

$$0 = P(z) := z e^{z\delta} - g_+(z) - g_-(z) \quad , \tag{8}$$

where $g_\pm$ admits the continued fraction representation

$$g_\pm(z) := \cfrac{\beta^2}{(z \pm i) e^{(z \pm i)\delta} - \cfrac{\beta^2}{(z \pm 2i) e^{(z \pm 2i)\delta} - \cfrac{\beta^2}{\cdots}}} \quad . \tag{9}$$

For recent applications of such expansions within the context of delayed feedback control the reader should also consult [14].

Its quite straightforward to estimate that the continued fraction is analytic in $z$ for $\mathrm{Re}(z)$ sufficiently large and that the inequality

$$|g_\pm(z)| \leq \frac{2\beta^2}{|\mathrm{Re}(z)| e^{\,\mathrm{Re}(z)\delta}} \tag{10}$$

holds provided that $z$ obeys

$$|\mathrm{Re}(z)| e^{\,\mathrm{Re}(z)\delta} > |2\beta| \quad . \tag{11}$$

Together with the characteristic equation (8) we conclude that there does not exist any eigenvalue in the complex half plane specified by the inequality (11) and that the latter condition considered as an equality yields an analytical upper bound for the spectrum.

The imaginary part of Floquet exponents is as usual defined modulo the external frequency. Within our notation (7) it means that the solutions of the characteristic equation (8) have to appear with period–one in the imaginary part. Expression (8) shares this property. Applying the recursion relations

$$g_+(z) = \frac{\beta^2}{(z+i) e^{(z+i)\delta} - g_+(z+i)} \tag{12}$$

$$g_-(z+i) = \frac{\beta^2}{z e^{z\delta} - g_-(z)} \tag{13}$$

which are an immediate consequence of the continued fraction representation (9) the characteristic equation obeys

$$P(z+i)=\frac{g_-(z+i)}{g_+(z)}P(z) \quad . \tag{14}$$

Eqs.(8) and (9) are sufficient to compute eigenvalues numerically by root finding schemes and, in particular, to monitor their variation with the two remaining relevant parameters, $\beta$ the dimensionless driving amplitude, and $\delta$ the ratio between delay and external period. Such a direct approach yields local information about the spectrum but is not able to capture some global features. In particular, no upper bound on the spectrum can be obtained in such a way. The latter goal may be achieved by contour integration and applying some features of meromorphic functions [15]. Roughly speaking, the number of zeros minus the number of poles of a complex function which occur within a connected region of the complex plane can be counted by observing how often the complex argument of the function winds around during a complete itinerary along the boundary of the region under consideration. Unfortunately, one would require an contour of infinite extent to apply the just mentioned idea since the characteristic equation (8) is not periodic along the imaginary axis. Hence, one would like to modify the characteristic equation (8) in such a way that it becomes a periodic function without changing of course the eigenvalues.

There exist no unique way to achieve our goal, although the main idea is quite simple. In view of the property (14) one needs to compensate for some factors in order to ensure periodicity. On a formal level such a step can be achieved if one includes two infinite products in the characteristic equation, $\prod_{k=1}^{\infty} g_+(z-ik)$ and $\prod_{k=1}^{\infty} g_-(z+ik)$. Unfortunately, one could not expect the products to converge, since the representation (9) indicates the asymptotic behaviour

$$g_\pm[z] \overset{z\to\infty}{\simeq} \frac{\beta^2}{(z\pm i)e^{(z\pm i)\delta}} \quad . \tag{15}$$

If one compensates for this asymptotics and introduces the quantity

$$h_\pm(z):=\frac{1}{\beta^2}(z\pm i)e^{(z\pm i)\delta}g_\pm(z) \tag{16}$$

then the expression

$$O(z):=\prod_{k=0}^{\infty} h_+(z+ik)\times\frac{ze^{z\delta}}{P(z)}\times\prod_{k=0}^{\infty} h_-(z-ik) \tag{17}$$

is well defined since the products may converge absolutely. Furthermore, the quantity obeys the periodicity constraint. Taking eqs.(14) and (16) into account one obtains

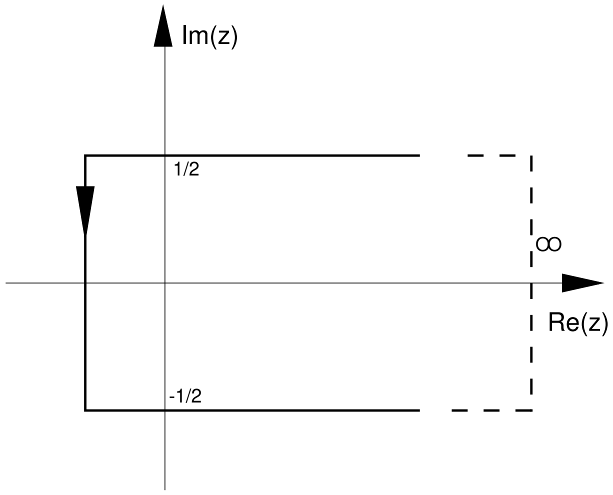


Fig. 1. Integration contour for the evaluation of the characteristic equation.

$$O(z+i) = \prod_{k=0}^{\infty} h_+(z+ik) \times \frac{1}{h_+(z)} \times \frac{(z+i)e^{(z+i)\delta} g_+(z)}{P(z) g_-(z+i)} \times h_-(z+i)$$
$$\times \prod_{k=0}^{\infty} h_-(z-ik) = O(z) \quad . \tag{18}$$

Except for an additional zero at $z=0$ (respectively at $z=in$, $n \in \mathbb{Z}$) the second factor in definition (17) causes poles at the solutions of the characteristic equation. As for the two infinite products I do not have rigorous proofs. But heuristic inspection of the definitions (16) and (9) together with numerical confirmation suggests that these terms remain regular and nonzero, at least if $\mathrm{Re}(z)$ is not too small. Taking this crucial assumption as granted, the poles of eq.(17) yield the desired eigenvalues. Because of the periodicity (18) these poles can be counted by contour integration as stressed above. For the contour displayed in figure 1 the contributions from the horizontal segments drop by the periodicity (18) and the right hand vertical segment does not contribute since the infinite products tend to unity in the limit of $\mathrm{Re}(z) \to \infty$. Depending on whether the left hand branch is located in the region $\mathrm{Re}(z) > 0$ or $\mathrm{Re}(z) < 0$, the number of eigenvalues respectively the number of eigenvalues minus one which are located right to the left hand branch is determined by counting how often $\arg(O(z))$ winds along the left hand branch. Summarising, the winding number of $\arg(O(z))$ along a finite segment determines how many eigenvalues are located to the right hand side of the segment. Finally, let me mention that from the point of view of its analytical structure expression (17) resembles dynamical zeta functions [16], but no deeper relationship seems to be available at the moment.

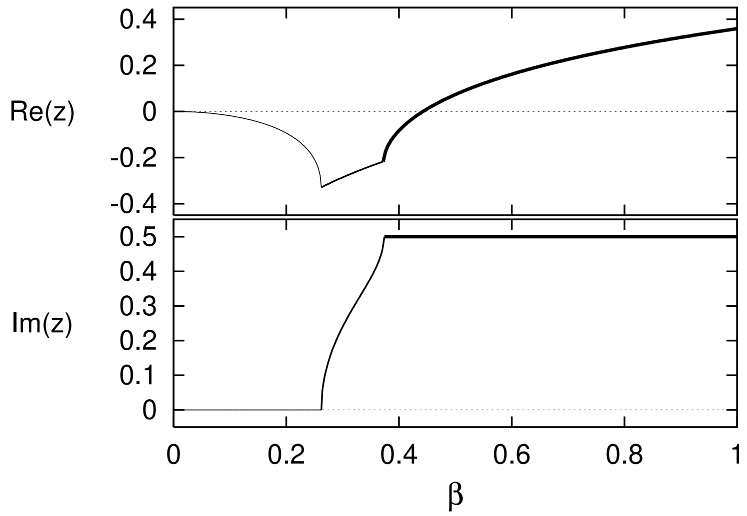


Fig. 2. Dependence of the real and imaginary part of the rescaled eigenvalue $z$ on the dimensionless driving amplitude $\beta$ for $\delta = 2\pi/3$. Different thickness of the lines correspond to different values of the imaginary part. For simplicity only one branch of the complex conjugated pair is displayed.

## 3 Numerical evaluation of the largest eigenvalue branch

In what follows we are mainly interested in the stability properties of our equation and focus therefore on the eigenvalue with largest real part. Within our dimensionless notation (7) the stability criterion reads $\mathrm{Re}(z) \leq -c/\omega$. It depends of course explicitly on the coefficient of the non–delayed part.

The eigenvalue with largest real part is computed from eq.(8) with standard root finding algorithms and continuation techniques in the parameters $\beta$ respectively $\delta$. The continued fractions (9) have been evaluated with Euler's summation formula (cf. appendix C.1) which gives numerically stable results and allows for error control. For the test on maximal real part the function (17) has been evaluated along the contour described above, and the products have been computed numerically by a stable simple forward iteration scheme (cf. appendix C.2).

First consider the behaviour for fixed value of $\delta$. In particular, let me focus on values where the period $T$ equals an integer multiple of the delay $\tau$, i. e. $\delta = 2\pi/n$, $n \in \mathbb{N}$. According to the discussion in the introduction such cases appear within the context of rhythmic control and are therefore of special interest. As already indicated the choices $n = 1$ and $n = 2$ lead to the trivial result $z = 0$ (cf. appendix A.2). Nontrivial features appear at $n = 3$ for the first time, and the dependence of $z$ on $\beta$ is shown in figure 2. On increasing $\beta$ the real eigenvalue $z$ decreases. At some finite value two real eigenvalues collide giving rise to a complex conjugated pair. As a consequence a finite imaginary part

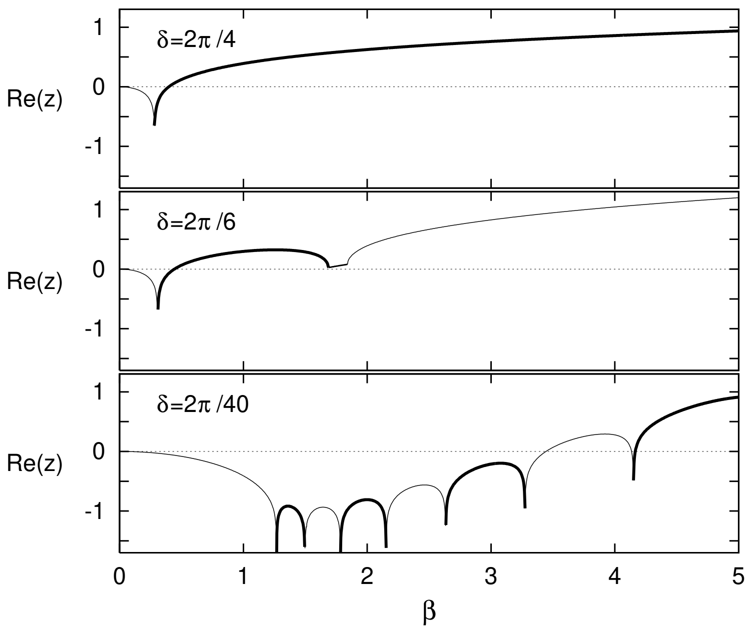


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Fig. 3. Real part of the rescaled eigenvalue $z$ on $\beta$ for three values of $\delta$. Different thickness of the lines correspond to different values of the imaginary part (cf. figure 2)

develops. That pair collides again at even higher driving amplitudes resulting in a pair of real negative multipliers, so that two exponents with imaginary part 1/2 are born. The multiplier with larger modulus governs the stability and the real part of the corresponding exponent increases and finally changes sign. From the point of view of stability orbits whose value of $-c/\omega$ is larger than than the minimum of the graph can become stable. The interval of stable $\beta$ values is limited by a transcritical bifurcation to the left and by a hopf or flip[2] bifurcation to the right.

Similar features appear for smaller values of the ratio $\delta$. Three special cases are displayed in figure 3. On increasing $\delta$ additional collisions of eigenvalues are created which finally may result in broader or even more than one stability interval. The price one has to pay is that these intervals shift to larger values of $\beta$ so that for $\delta \to 0$ one ends up again with the single eigenvalue $z = 0$.

In order to understand the just mentioned features more systematically let me investigate the eigenvalue in the two–dimensional parameter plane without restricting to some integer relation between delay and period. Figure 4 contains the corresponding data within the range $\delta \in [\pi, 2\pi]$. As already stressed the real part vanishes at $\delta = \pi$ and $\delta = 2\pi$. In between $\mathrm{Re}(z)$ is positive and the imaginary part vanishes within the whole range. Nothing significant appears from the point of view of stabilisation. Things are quite different in the region

[2] Often such a bifurcation is also called period doubling bifurcation.

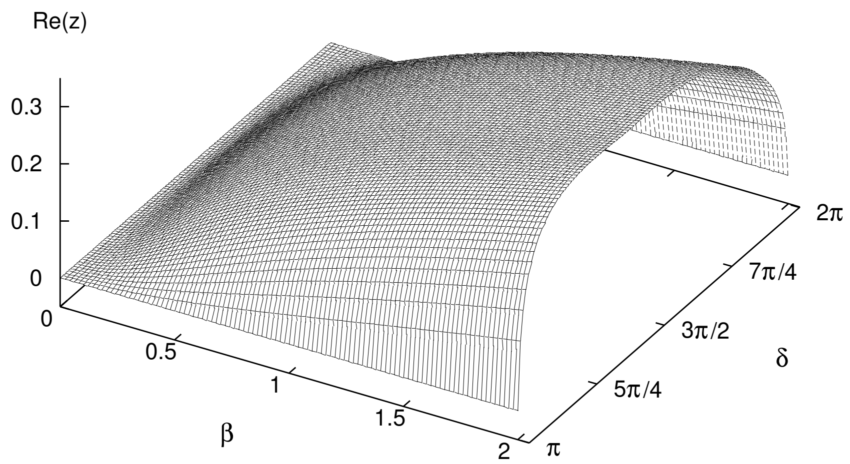
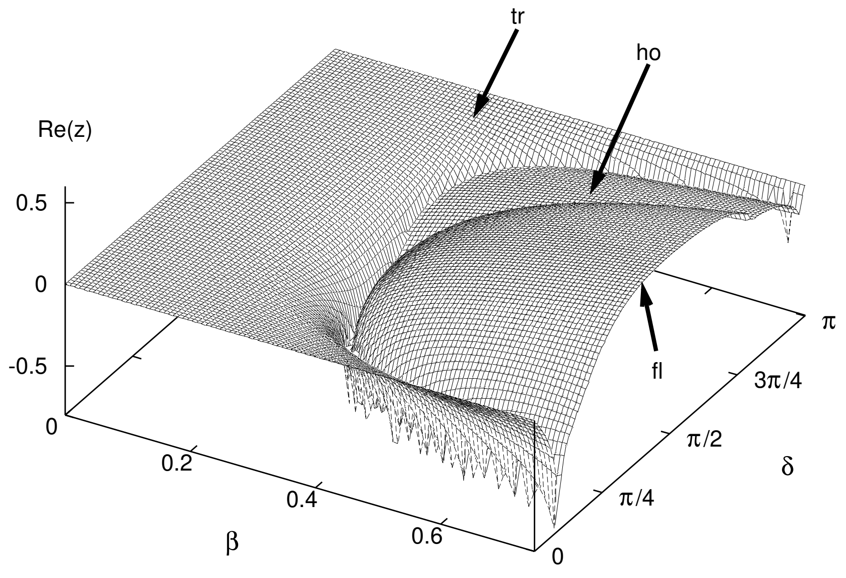



Fig. 4. Real part of the rescaled eigenvalue $z$ in dependence on the dimensionless driving amplitude $\beta$ and the ratio of period and delay $\delta$. $\mathrm{Im}(z) \equiv 0$ on the whole surface.

of small $\delta$ (cf. figure 5). Here the real part may become negative, although for increasing $\beta$–values $\mathrm{Re}(z)$ finally increases and attains positive values. Regions with finite imaginary part and large domains with negative multipliers ($\mathrm{Im}(z) = 1/2$) are visible beyond certain threshold values of the driving amplitude. In the vicinity of $\delta = 0$ the structure of the eigenvalues become quite intricate. Banana shaped domains with imaginary part 0 and 1/2 alternate, which are separated by deep valleys in the real part. From these structures the slices at fixed value of $\delta$ (cf. figures 2 and 3) are easily recovered. Although the eigenvalue tends to zero for $\delta \to 0$ the limit is apparently non–uniform in $\beta$.

## 4 Conclusion

The analysis has shown that the eigenvalues of time–delayed Floquet problems behave quite different compared to the case of autonomous time delay systems. The competition between the two different time scales, delay and the external period drastically influences the dependence of the largest eigenvalue on the driving amplitude. In particular, collisions of eigenvalues trigger the decrease of the real part which may lead to domains of stable solutions. The dependence becomes quite intricate if the ratio between delay and external period becomes



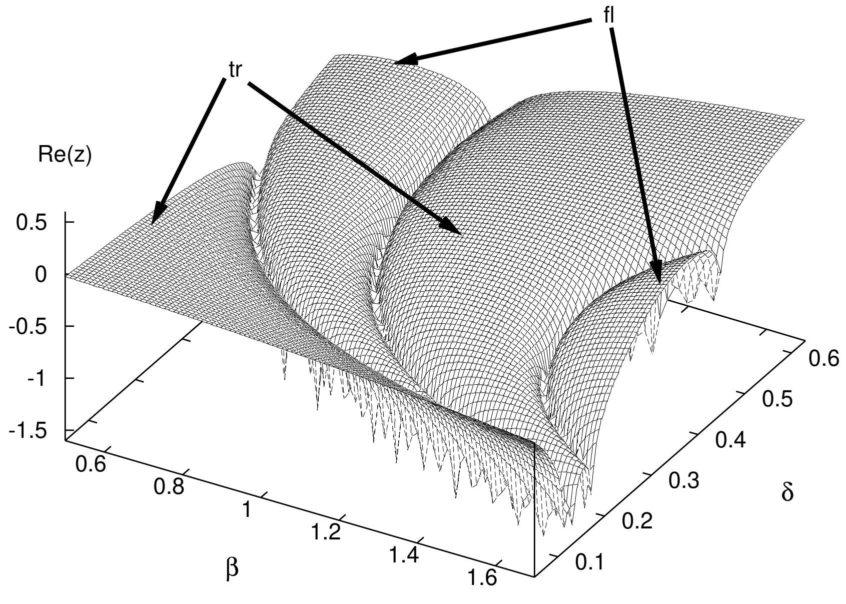




Fig. 5. Real part of the rescaled eigenvalue $z$ in dependence on on the dimensionless driving amplitude $\beta$ and the ratio of period and delay $\delta$. Symbols indicate the value of the imaginary part on different parts of the surface, tr: $\mathrm{Im}(z) = 0$, fl: $\mathrm{Im}(z) = 1/2$, ho: $\mathrm{Im}(z) \in (0, 1/2)$.

small. Such features have implications for the stabilisation of periodic states by rhythmic control since an improvement is suggested if large modulation periods are employed.

The explicit calculation has been performed on a simple scalar model equation with harmonic time dependence. Therefore the computations and the closed anlytical result for the characteristic equation are quite simple. However, the general approach is not limited to this case and extensions to anharmonic driving fields or vector valued equations (cf. [13,14]) are straithforeward. Of course the resulting continued fraction may become more intricate.

In some respect the applied expansion seems to be very suitable for studying time delay systems. Since the delay already causes a formally infinite dimensional dynamical system even simple models giving rise to scalar valued continued fractions show rather complicated spectra. On the other hand the expansion is simple and appropriate for accurate numerical evaluation so that all details of the spectra are accesible. Since a closed expression for the characteristic equation is available several features, e. g. the limit of small $\delta$ and large $\beta$ may be studied analytically by asymptotic expansions.

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## A Transformation properties

### *A.1 Floquet transformation*

Consider the general time–dependent linear differential–difference equation

$$\dot{y}(t) = R(t)y(t) + S(t)y(t-\tau) \quad , \tag{A.1}$$

where the coefficients $R$ and $S$ are periodic functions of $t$ with period $T$ and all quantities may be vector respectively matrix valued. The periodic time dependence of $R$ can be eliminated taking the Floquet decomposition of the non–delayed part into account. To be definite, let $U(t)$ denote the evolution matrix of the linear equation,

$$\dot{U}(t) = R(t)U(t), \quad U(0) = 1 \quad . \tag{A.2}$$

By virtue of the Floquet theory it admits the decomposition

$$U(t) = Q(t)\exp(Ct), \quad Q(t) = Q(t+T) \quad , \tag{A.3}$$

where the periodic factor $Q$ is invertible. Introducing new variables

$$y(t) =: Q(t)z(t) \tag{A.4}$$

eq.(A.1) reads

$$\begin{aligned} \dot{z}(t) &= \left[Q^{-1}(t)R(t)Q(t) - Q^{-1}(t)\dot{Q}(t)\right] z(t) + Q^{-1}(t)S(t)Q(t)z(t-\tau) \\ &= Cz(t) + Q^{-1}(t)S(t)Q(t)z(t-\tau) \quad . \end{aligned} \tag{A.5}$$

The last equality is an immediate consequence of eqs.(A.3) and (A.2). Hence, it is sufficient to consider time–independent coefficients $R$, since the general case (A.1) follows from a simple linear transformation. Of course such a feature is well known and we have included the discussion for completeness only.

### *A.2 Reduction to ordinary differential equations*

Suppose the period $T$ and the delay $\tau$ are rationally related i. e. that

$$T = N\Delta t, \quad \tau = m\Delta t, \quad m, N \in \mathbb{N} \tag{A.6}$$

holds for appropriate values of $\Delta t$, $m$, $N$. Introducing

$$\begin{aligned} p_n(t') &:= q(n\Delta t + t') \\ A_n(t') &:= B(n\Delta t + t'), \quad t' \in [0, \Delta t], \quad 0 \le n \le N-1 \end{aligned} \tag{A.7}$$

eq.(5) is cast into the form

$$\lambda p_n(t') + \dot{p}_n(t') = cp_n(t') + e^{-\lambda \Delta t m} A_n(t') p_{n-m}(t') \quad , \tag{A.8}$$

where from now on all indices are considered modulo $N$. By virtue of the definition (A.7) $p_n$ obeys the boundary condition

$$p_{n+1}(0) = p_n(\Delta t), \quad 0 \le n \le N-1 \quad . \tag{A.9}$$

There exists a standard approach (cf. [11]) to compute the characteristic equation for the multipliers

$$\mu := e^{-\lambda \Delta t} \tag{A.10}$$

from the ordinary boundary value problem (A.8), (A.9). Let $\boldsymbol{U}$ denote the evolution matrix of the system corresponding to eq.(A.8), i. e.

$$\dot{U}_{i,j}(t') = cU_{i,j}(t') + \mu^m A_i(t') U_{i-m,j}(t'), \quad U_{i,j}(0) = \delta_{i,j} \quad . \tag{A.11}$$

The boundary condition (A.9) tells us that the eigensolution is obtained from the eigenvector of $\boldsymbol{Q}\,\boldsymbol{U}(\Delta t)$ with eigenvalue $\mu$, where

$$Q_{i,j} := \delta_{i-1,j} \tag{A.12}$$

performs the cyclic permutation of the indices. Hence, the eigenvalues are determined by the condition

$$0 = \det\left(\,\mathbf{1} - \mu\,\boldsymbol{Q}\,\boldsymbol{U}(\Delta t)\right) \tag{A.13}$$

and the numerical evaluation just requires the integration of the set of ordinary differential equations (A.11). In addition, the analytic dependence on $\mu$ allows for obtaining upper bounds for the spectrum by integration over finite contours.

The full analytical discussion of eq.(A.11) is of course difficult to perform and a systematic trend with the variation of the period or the delay is difficult to obtain since the dimension changes with these parameters. However, special cases can be analysed even analytically. Let us consider the case where the period $T$ equals twice the delay $\tau$, that means $m = 1$, $N = 2$ in eq.(A.6). For the choice (2) the system (A.11) reads

$$\dot{\boldsymbol{U}}(t') = \begin{pmatrix} c & \cos(\omega t') \\ -\cos(\omega t') & c \end{pmatrix} \boldsymbol{U}(t') \quad , \tag{A.14}$$

and plain integration yields

$$\boldsymbol{U}_{i,j}(\Delta t) = e^{cT/2}\delta_{i,j} \quad . \tag{A.15}$$

Finally the characteristic equation (A.13) results in

$$1 = e^{(c-\lambda)T} \quad . \tag{A.16}$$

Hence, only one Floquet exponent $\lambda = c$ occurs like in the trivial cases $T = \tau$ or $B \equiv 0$. The result does not depend on the particular choice (2) but requires some antisymmetry of the time–dependent coefficient.

## B Continued fraction expansion

Using the Fourier representation

$$q(t) = \sum_{n=-\infty}^{\infty} q_n e^{in\omega t} \tag{B.1}$$

the eigenvalue problem (5), (2) is written as

$$0 = (z + in)e^{(z+in)\delta}\alpha_n - \beta\alpha_{n-1} - \beta\alpha_{n+1} \quad , \tag{B.2}$$

where the abbreviations (7) have been used and a phase factor has been absorbed in the Fourier coefficients

$$\alpha_n := e^{-in\delta} q_n \quad . \tag{B.3}$$

On the one hand eq.(B.2) yields for $n = 0$

$$0 = ze^{z\delta} - \beta\frac{\alpha_{-1}}{\alpha_0} - \beta\frac{\alpha_1}{\alpha_0} \quad . \tag{B.4}$$

On the other hand the same equation can be cast into the form of two equivalent iteration schemes as

$$\beta\frac{\alpha_n}{\alpha_{n-1}} = [z + i(n-1)]\, e^{[z+i(n-1)]\delta} - \frac{\beta^2}{\beta\alpha_{n-1}/\alpha_{n-2}} \tag{B.5}$$

$$\beta\frac{\alpha_n}{\alpha_{n+1}} = [z + i(n+1)]\, e^{[z+i(n+1)]\delta} - \frac{\beta^2}{\beta\alpha_{n+1}/\alpha_{n+2}} \quad . \tag{B.6}$$

Iterating eq.(B.5) for $n \leq 0$ respectively eq.(B.6) for $n \geq 0$ one obtains the continued fraction representations (9), and eq.(B.4) yields eq.(8).

There is nothing special about the index $n = 0$ and one may base the scheme at any other fixed value $n_0$. The resulting eigenvalue equation differs from eq.(8) by shifting the argument $z$ to $z + in_0$, $n_0 \in \mathbb{Z}$. In view of the periodicity property (14) the result for the eigenvalues remains completely unchanged and it is sufficient to consider one of these characteristic equations only.

## C Evaluation of continued fractions and products

### *C.1 Euler's summation formula*

For the continued fraction

$$R := \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \cfrac{a_3}{\cdots}}} \tag{C.1}$$

the $n$-th convergent, i.e. the expression which results by setting $a_{n+1} = 0$ is given by [17]

$$R_n = \sum_{k=1}^{n} \prod_{j=1}^{k} \rho_j \quad , \tag{C.2}$$

where

$$\begin{aligned} \rho_1 &= a_1/b_1 \\ 1+\rho_2 &= \frac{1}{1+a_2/(b_1 b_2)} \\ 1+\rho_k &= \frac{1}{1+(1+\rho_{k-1})a_k/(b_k b_{k-1})}, \quad k \geq 3 \quad . \end{aligned} \tag{C.3}$$

### *C.2 Evaluation of infinite products*

From the definition (16) and the property (14) one obtains for the finite product

$$\prod_{k=0}^{N-1} h_\pm(z \pm ik) = \frac{P(z)}{ze^{z\delta}} \times \prod_{k=1}^{N} h_\mp(z \pm ik) \times \frac{(z \pm iN)e^{(z \pm iN)\delta}}{P(z \pm iN)} \quad . \tag{C.4}$$

In view of the asymptotics (15) the last factor tends to one and we obtain

$$\prod_{k=0}^{\infty} h_\pm(z \pm ik) = \frac{P(z)}{ze^{z\delta}} \times \prod_{k=1}^{\infty} h_\mp(z \pm ik) \quad . \tag{C.5}$$

The factors in the product on the right hand side can be obtained in a numerically stable way by simple iteration. In fact, taking the definition (16) and the recusion properties (12), (13) into account simple algebra yields

$$h_\pm(z \mp i(k+1)) = \cfrac{1}{1 - \cfrac{\beta^2 h_\pm(z \mp ik)}{(z \mp ik)e^{(z \mp ik)\delta}(z \mp i[k-1])e^{(z \mp i[k-1])\delta}}} \quad . \tag{C.6}$$

The numerical evaluation of eqs.(C.6) and (C.4) is now straightforward.

One should note that the product on the right hand side of eq.(C.5) cancels the pole and the zeros of the first factor.

## D Consistency check

In the trivial case

$$\delta = 2\pi n, \quad n \in \mathbb{Z} \tag{D.1}$$

one can easily sum the continued fraction (9) in closed analytical form [17] since the exponentials do not depend on $\delta$. Such a resummation is useful for a consistency check of the analytical properties of our quantities and I dwell on this subject in what follows.

Using a simple generalised hypergeometric series

$${}_0F_1(a,x) = 1 + \frac{x}{a} + \frac{x^2}{a(a+1)2!} + \dots \tag{D.2}$$

and its recurrence relation

$$\frac{{}_0F_1(a,x)}{{}_0F_1(a+1,x)} = 1 + \frac{x}{a(a+1)} \frac{{}_0F_1(a+2,x)}{{}_0F_1(a+1,x)} \tag{D.3}$$

the continued fraction (9) is written as (cf. eqs.(12) and (13))

$$\begin{aligned} g_\pm(z) &= z e^{z\delta} \left[ 1 - \frac{{}_0F_1\left(\mp iz, \beta^2 e^{-2z\delta}\right)}{{}_0F_1\left(\mp iz + 1, \beta^2 e^{-2z\delta}\right)} \right] \\ &= z e^{z\delta} \left( - \frac{\beta^2 e^{-2z\delta}}{\mp iz(\mp iz + 1)} \right) \frac{{}_0F_1\left(\mp iz + 2, \beta^2 e^{-2z\delta}\right)}{{}_0F_1\left(\mp iz + 1, \beta^2 e^{-2z\delta}\right)} \quad . \end{aligned} \tag{D.4}$$

Then the characteristic equation (8) reads

$$\begin{aligned} P(z) = z e^{z\delta} \Bigg[ & 1 + \frac{\beta^2 e^{-2z\delta}}{(-iz)(-iz+1)} \frac{{}_0F_1\left(-iz + 2, \beta^2 e^{-2z\delta}\right)}{{}_0F_1\left(-iz + 1, \beta^2 e^{-2z\delta}\right)} \\ & + \frac{\beta^2 e^{-2z\delta}}{iz(iz+1)} \frac{{}_0F_1\left(iz + 2, \beta^2 e^{-2z\delta}\right)}{{}_0F_1\left(iz + 1, \beta^2 e^{-2z\delta}\right)} \Bigg] \quad . \end{aligned} \tag{D.5}$$

Since the hypergeometric series is regular at $z = 0$ the second factor remains finite and the eigenvalue $z = 0$ is easily recovered. The corresponding integer multiples, $z = ik$, $k \in \mathbb{Z}$ are caused by the second factor, as easily computed from the singularities of the hypergeometric series (D.2). No additional solution is apparent.

As for the infinite products in eq.(17) the quantity (16) reads, taking the expression (D.4) into account

$$h_\pm(z) = \frac{{}_0F_1(\mp iz + 2, \beta^2 e^{-2z\delta})}{{}_0F_1(\mp iz + 1, \beta^2 e^{-2z\delta})} \quad . \tag{D.6}$$

The products have telescopic form and yield

$$\prod_{k=0}^{\infty} h_\pm(z \pm ik) = \frac{1}{{}_0F_1\left(\mp iz + 1, \beta^2 e^{-2z\delta}\right)} \quad , \tag{D.7}$$

where convergence follows from

$$\lim_{k\to\infty} {}_0F_1\left(\mp iz + k, \beta^2 e^{-2z\delta}\right) = 1 \quad . \tag{D.8}$$

Finally, taking eqs.(D.5) and (D.7) into account eq.(17) results in

$$\begin{aligned} O(z) = &\Big[{}_0F_1\left(-iz + 1, \beta^2 e^{-2z\delta}\right) {}_0F_1\left(iz + 1, \beta^2 e^{-2z\delta}\right) \\ &+ \frac{\beta^2 e^{-2z\delta}}{-iz(-iz+1)} {}_0F_1\left(-iz + 2, \beta^2 e^{-2z\delta}\right) {}_0F_1\left(iz + 1, \beta^2 e^{-2z\delta}\right) \\ &+ \frac{\beta^2 e^{-2z\delta}}{iz(iz+1)} {}_0F_1\left(iz + 2, \beta^2 e^{-2z\delta}\right) {}_0F_1\left(-iz + 1, \beta^2 e^{-2z\delta}\right)\Big]^{-1} \\ = &1 \quad . \end{aligned} \tag{D.9}$$

The final result follows from a nontrivial identity of hypergeometric functions. In this degenerated case the poles at $z = ik$, $k \in \mathbb{Z}$ cancel exactly with the additional zeros at the same positions.

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